

DORFMAN CONNECTIONS AND COURANT ALGEBROIDS

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ABSTRACT. A linear connection $\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ on a vector bundle E over a smooth manifold M is tantamount to a splitting $TE \xrightarrow{\sim} V \oplus H_\nabla$, where V is the set of vectors tangent to the fibres of E . Furthermore, the curvature of the connection measures the failure of the horizontal space H_∇ to be involutive. In this paper, we show that splittings $TE \oplus T^*E \xrightarrow{\sim} (V \oplus V^\circ) \oplus L$ of the Pontryagin bundle over the vector bundle E can be described in the same manner via a certain class of maps $\Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \rightarrow \Gamma(E \oplus T^*M)$. Similarly to the tangent case, we find that, after the choice of a splitting, the Courant algebroid structure of $TE \oplus T^*E \rightarrow E$ can be completely described by properties of the map Δ .

The maps Δ in this correspondence theorem are particular examples of connection-like maps that we define in this paper and name Dorfman connections. Roughly said, these objects are to Courant algebroids what connections are to Lie algebroids.

In a second part of this paper, we study splittings of $TA \oplus T^*A$ over a Lie algebroid A , and we show how \mathcal{LA} -Dirac structures on A are in bijection with a class of Manin triples over the base manifold M . This has as special cases the correspondences between Lie bialgebroids and Poisson algebroids, and between IM-2-forms and presymplectic algebroids.

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1. INTRODUCTION

Let us start with a simple observation. Take a subbundle $F \subseteq TM$ of the tangent space of a smooth manifold M . Then the \mathbb{R} -bilinear map

$$\begin{aligned}\tilde{\nabla} : \Gamma(F) \times \mathfrak{X}(M) &\rightarrow \Gamma(TM/F), \\ \tilde{\nabla}_f X &= \overline{[f, X]}\end{aligned}$$

measures the failure of vector fields on M to preserve F . The subbundle F is involutive if and only if $\tilde{\nabla}_f f' = 0$ for all $f, f' \in \Gamma(F)$, and in this case, $\tilde{\nabla}$ induces a flat connection

$$\begin{aligned}\nabla : \Gamma(F) \times \Gamma(TM/F) &\rightarrow \Gamma(TM/F), \\ \tilde{\nabla}_f \bar{X} &= \overline{[f, \bar{X}]},\end{aligned}$$

the **Bott connection** associated to F [2].

In the same manner, given a Courant algebroid $\mathbf{E} \rightarrow M$ with bracket $\llbracket \cdot, \cdot \rrbracket$, anchor ρ and pairing $\langle \cdot, \cdot \rangle$, and a subbundle $K \subseteq \mathbf{E}$, one can define an \mathbb{R} -bilinear map

$$\begin{aligned}\tilde{\Delta} : \Gamma(K) \times \Gamma(\mathbf{E}) &\rightarrow \Gamma(\mathbf{E}/K), \\ \tilde{\Delta}_k e &= \overline{\llbracket k, e \rrbracket}.\end{aligned}$$

Again, we have $\tilde{\Delta}_k k' = 0$ for all $k, k' \in \Gamma(K)$ if and only if K is a subalgebroid of \mathbf{E} . If K is in addition isotropic, it is a Lie algebroid over M and the pairing on \mathbf{E} induces a pairing $K \times_M (\mathbf{E}/K) \rightarrow \mathbb{R}$. The \mathbb{R} -bilinear map

$$\begin{aligned}\Delta : \Gamma(K) \times \Gamma(\mathbf{E}/K) &\rightarrow \Gamma(\mathbf{E}/K), \\ \tilde{\Delta}_k \bar{e} &= \overline{\llbracket k, e \rrbracket}\end{aligned}$$

that is induced by $\tilde{\Delta}$ is not a connection because it is not $C^\infty(M)$ -homogeneous in the first argument, but the obstruction to this is, as we will see, measured by the pairing, the anchor of the Courant algebroid and the de Rham derivative on $C^\infty(M)$. This map is an example of what is called in this paper a Dorfman connection, namely the **Bott–Dorfman connection** associated to K . In this paper, Dorfman connections are defined, and some of their properties and applications are studied. Dorfman connections appear naturally in several situations related to Courant algebroids and play a role similar to the one that connections play for tangent bundles and Lie algebroids.

It is for instance well-known that a linear TM -connection ∇ on a vector bundle $q_E : E \rightarrow M$ corresponds to a splitting $TE \xrightarrow{\sim} T^{q_E} E \oplus H_\nabla$. The failure of the horizontal space H_∇ to be involutive is measured by the curvature of the connection, and the connection itself is, roughly said, nothing else than the Lie bracket of horizontal and vertical vector fields, since it can be seen as a projection of the Bott connection

$$\nabla^{H_\nabla} : \Gamma(H_\nabla) \times \Gamma(TE/H_\nabla) \rightarrow \Gamma(TE/H_\nabla).$$

in a sense that we will explain.

The first main result of this paper is the one-to-one correspondence in the same spirit between splittings of the standard Courant algebroid over E as

$$TE \oplus T^*E \xrightarrow{\sim} (T^{q_E} E \oplus (T^{q_E} E)^\circ) \oplus L_\Delta$$

and $TM \oplus E^*$ -Dorfman connections Δ on $E \oplus T^*M$. The failure of L to be isotropic (and so Lagrangian) relative to the canonical pairing on $TE \oplus T^*E$ is equivalent to the failure of the dual of the Dorfman connection (in the sense of connections) to be antisymmetric, and the failure of L to be closed under the Courant algebroid bracket is measured by the curvature of the Dorfman connection. The Dorfman connection is the same as the Courant-Dorfman bracket on horizontal and vertical sections.

We characterize then double vector subbundles of $TE \oplus T^*E$ over E and a subbundle $U \subseteq TM \oplus E^*$. Such double vector bundles can be described by triples U, K and Δ , where Δ is a Dorfman connection and K a subbundle of $E \oplus T^*M$. For instance, we study Dirac structures on E that define double vector subbundles of the Pontryagin bundle. We prove that maximal isotropy and integrability of the Dirac structure depend only on (simple) properties of the corresponding triple.

If the vector bundle $E =: A$ has a Lie algebroid structure $(q_A : A \rightarrow M, \rho, [\cdot, \cdot])$, then the Pontryagin bundle $TA \oplus T^*A$ has a naturally induced Lie algebroid structure over $TM \oplus A^*$. Given a $TM \oplus A^*$ -Dorfman connection Δ on $A \oplus T^*M$, we compute the representation up to homotopy that corresponds to the splitting $TA \oplus T^*A \xrightarrow{\sim} (T^{q_A}A \oplus (T^{q_A}A)^*) \oplus L_\Delta$ and describes the VB-algebroid $TA \oplus T^*A \rightarrow TM \oplus A^*$ [10]. This representation up to homotopy is in general not the product of the two representations up to homotopy describing $TA \rightarrow TM$ and $T^*A \rightarrow A^*$.

Knowing this, one can ask when a Dirac structure on A , that is a double sub-vector bundle of $TA \oplus T^*A$, is at the same time a Lie subalgebroid of $TA \oplus T^*A \rightarrow TM \oplus A^*$ over its base $U \subseteq TM \oplus A^*$. In that case, the Dirac structure has the induced structure of a double Lie algebroid, and is called an **\mathcal{LA} -Dirac structure** on A . A known example of an \mathcal{LA} -Dirac structure on a Lie algebroid A is the graph of $\pi^\sharp : T^*A \rightarrow TA$, where π_A is the linear Poisson bivector field defined on A by a Lie algebroid structure on A^* such that (A, A^*) is a Lie bialgebroid [25]. In that case, we know that the \mathcal{LA} -Dirac structure is completely encoded in the Lie bialgebroid, which is itself equivalent to a Courant algebroid with two transverse Dirac structures [20].

The second standard example is that of the graph of a linear presymplectic form $\sigma^* \omega_{\text{can}} \in \Omega^2(A)$, where $\sigma : A \rightarrow T^*M$ is an IM-2-form [6, 5]. Here also, the \mathcal{LA} -Dirac structure is equivalent to the IM-2-form, and any \mathcal{LA} -Dirac structure that is the graph of a presymplectic form on A arises in this manner.

The final, most easy, example is $F_A \oplus F_A^\circ$, where $F_A \rightarrow A$ is an involutive subbundle that has at the same time a Lie algebroid structure over some subbundle $F_M \subseteq TM$. Here we know that the \mathcal{LA} -Dirac structure corresponds to an infinitesimal ideal system, i.e. a triple (C, F_M, ∇) , where $C \subseteq A$ is a subalgebroid, $F_M \subseteq TM$ and involutive subbundle and $\nabla : \Gamma(F_M) \times \Gamma(A/C) \rightarrow \Gamma(A/C)$ a flat connection satisfying some properties that allow ones to make sense of a quotient Lie algebroid $(A/C)/\nabla \rightarrow M/F_M$ [16].

The second main result of this paper is a classification of \mathcal{LA} -Dirac structures on Lie algebroids via a certain class of Manin pairs [7] on the base space of the Lie algebroid. This result unifies the three correspondences summarized above.

Outline of the paper. Some background on Courant algebroids and Dirac structures, connections, and double vector bundles is collected in the first section. In the second section, Dorfman sections and dull algebroids are defined, and examples are listed. In the third section, splittings of the standard Courant algebroid $TE \oplus T^*E$ over a vector bundle E are shown to be equivalent to a certain class of $TM \oplus E^*$ -Dorfman connections on $E \oplus T^*M$. Linear Dirac structures on the vector bundle $E \rightarrow M$ are studied via Dorfman connections. In the fourth section, the geometric structures on the two sides of the standard \mathcal{LA} -Courant algebroid $TA \oplus T^*A$ over a Lie algebroid $A \rightarrow M$ are expressed via splittings of $TA \oplus T^*A$, and \mathcal{LA} -Dirac structures on A are classified via Dorfman connections and some adequate

vector bundles over the units M . Finally, it is shown that this data is equivalent to a Manin pair over M , that is in a sense compatible with the Lie algebroid A .

Notation and conventions. Let M be a smooth manifold. We will denote by $\mathfrak{X}(M)$ and $\Omega^1(M)$ the spaces of (local) smooth sections of the tangent and the cotangent bundle, respectively. For an arbitrary vector bundle $E \rightarrow M$, the space of (local) sections of E will be written $\Gamma(E)$. We write $p_M : TM \rightarrow M$, $c_M : T^*M \rightarrow M$ and $\pi_M : TM \oplus T^*M \rightarrow M$ for the canonical projections, and $q_E : E \rightarrow M$ for vector bundle maps.

The flow of a vector field $X \in \mathfrak{X}(M)$ will be written ϕ^X , unless specified otherwise. Let $f : M \rightarrow N$ be a smooth map between two smooth manifolds M and N . Then two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be **f -related** if $Tf \circ X = Y \circ f$ on $\text{Dom}(X) \cap f^{-1}(\text{Dom}(Y))$. We write then $X \sim_f Y$.

Given a section ξ of E^* , we will always write $\ell_\xi : E \rightarrow \mathbb{R}$ for the linear function associated to it, i.e. the function defined by $e_m \mapsto \langle \xi(m), e_m \rangle$ for all $e_m \in E$. Given a section $e \in \Gamma(E)$, we write $e^\uparrow \in \mathfrak{X}(E)$ for the derivation along e , i.e. the vector field with complete flow $\mathbb{R} \times E \rightarrow E$, $(t, e'_m) \mapsto e'_m + te(m)$. Note that the vector fields e^\uparrow , for all $e \in \Gamma(E)$, span the subbundle $V := T^{q_E}E \subseteq TE$ and e^\uparrow is completely determined by $e^\uparrow(\ell_\xi) = q_E^*\langle \xi, e \rangle$ and $e^\uparrow(q_E^*\varphi) = 0$ for all $\varphi \in C^\infty(M)$ and $\xi \in \Gamma(E^*)$. Note furthermore that $e^\uparrow(f_m)$ depends only on $e(m)$ and f_m , so the expression $(e(m))^\uparrow(f_m)$ makes sense. In the same manner, given a vector bundle morphism $\phi \in \Gamma(\text{Hom}(E; E))$, we can define $\phi^\uparrow \in \mathfrak{X}(E)$, $\phi^\uparrow(e_m) = (\phi(e_m))^\uparrow(e_m)$ for all $e_m \in E$.

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2. BACKGROUND DEFINITIONS

We recall first some necessary background notions on Courant algebroids, of the double vector bundle structures on the tangent and cotangent spaces TE and T^*E of a vector bundle E , and on connections.

2.1. Courant algebroids and Dirac structures. A Courant algebroid [20, 27] over a manifold M is a vector bundle $E \rightarrow M$ equipped with a fibrewise nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a bilinear bracket $[\cdot, \cdot]$ on the smooth sections $\Gamma(E)$, and a vector bundle map $\rho : E \rightarrow TM$ over the identity called the anchor, which satisfy the following conditions

- (1) $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$,
- (2) $\rho(e_1)\langle e_2, e_3 \rangle = \langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle$,
- (3) $[e_1, e_2] + [e_2, e_1] = \mathcal{D}\langle e_1, e_2 \rangle$

for all $e_1, e_2, e_3 \in \Gamma(E)$ and $\varphi \in C^\infty(M)$. Here, we use the notation $\mathcal{D} := \rho^* \circ d : C^\infty(M) \rightarrow \Gamma(E)$, using $\langle \cdot, \cdot \rangle$ to identify E with E^* :

$$\langle \mathcal{D}\varphi, e \rangle = \rho(e)(\varphi)$$

for all $\varphi \in C^\infty(M)$ and $e \in \Gamma(E)$. The following conditions

- (4) $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)]$,
- (5) $[e_1, \varphi e_2] = \varphi[e_1, e_2] + (\rho(e_1)\varphi)e_2$

are then also satisfied. They are often part of the definition in the literature, but it was already observed in [28] that they follow from (1) – (3).¹

¹Actually, they both follow immediately from (2). To get (4) replace e_2 by φe_2 in (2), and to get (5), replace e_1 by $[e_1, e'_1]$: an easy computation yields then that $\rho[e_1, e'_1]\langle e_2, e_3 \rangle = [\rho(e_1), \rho(e'_1)]\langle e_2, e_3 \rangle$ for all $e_2, e_3 \in \Gamma(E)$.

Example 2.1. The direct sum $TM \oplus T^*M$ endowed with the projection on TM as anchor map, $\rho = \text{pr}_{TM}$, the symmetric bracket $\langle \cdot, \cdot \rangle$ given by

$$(2.1) \quad \langle (v_m, \alpha_m), (w_m, \beta_m) \rangle = \alpha_m(w_m) + \beta_m(v_m)$$

for all $m \in M$, $v_m, w_m \in T_m M$ and $\alpha_m, \beta_m \in T_m^* M$ and the **Courant-Dorfman bracket** given by

$$(2.2) \quad [(X, \alpha), (Y, \beta)] = ([X, Y], \mathcal{L}_X \beta - \mathbf{i}_Y \mathbf{d}\alpha)$$

$$(2.3)$$

for all $(X, \alpha), (Y, \beta) \in \Gamma(TM \oplus T^*M)$, yield the standard example of a Courant algebroid (often called the *standard Courant algebroid over M*). The map $\mathcal{D} : C^\infty(M) \rightarrow \Gamma(TM \oplus T^*M)$ is given by $\mathcal{D}f = (0, \mathbf{d}f)$.

A **Dirac structure** $\mathbf{D} \subseteq \mathbf{E}$ is a subbundle satisfying

- (1) $\mathbf{D}^\perp = \mathbf{D}$ relative to the pairing on \mathbf{E} ,
- (2) $[\Gamma(\mathbf{D}), \Gamma(\mathbf{D})] \subseteq \Gamma(\mathbf{D})$.

The rank of the Dirac bundle \mathbf{D} is then half the rank of \mathbf{E} , and the triple $(\mathbf{D} \rightarrow M, \rho|_{\mathbf{D}}, [\cdot, \cdot]|_{\Gamma(\mathbf{D}) \times \Gamma(\mathbf{D})})$ is a Lie algebroid on M . Dirac structures appear naturally in several contexts in geometry and geometric mechanics (see for instance [3] for an introduction to the geometry and applications of Dirac structures).

2.2. The double vector bundles TE and T^*E . Consider a vector bundle $q_E : E \rightarrow M$. Then the tangent space TE of E has two vector bundle structures. First, the usual vector bundle structure of the tangent space, $p_E : TE \rightarrow E$ and second the vector bundle structure $Tq_E : TE \rightarrow TM$, with the addition defined as follows. If x_{e_m} and $x_{e'_m}$ are such that $Tq_E(x_{e_m}) = Tq_E(x_{e'_m}) =: x_m \in TM$, then there exist curves $c, c' : (-\varepsilon, \varepsilon) \rightarrow E$ such that $\dot{c}(0) = x_{e_m}$, $\dot{c}'(0) = x_{e'_m}$ and $q_E \circ c = q_E \circ c'$. The sum $x_{e_m} +_{Tq_E} x_{e'_m}$ is then defined as

$$x_{e_m} +_{Tq_E} x_{e'_m} = \left. \frac{d}{dt} \right|_{t=0} c(t) +_{q_E} c'(t) \in T_{e_m + e'_m} E.$$

We get a double vector bundle

$$\begin{array}{ccc} TE & \xrightarrow{p_E} & E \\ Tq_E \downarrow & & \downarrow q_E \\ TM & \xrightarrow{p_M} & M \end{array} ,$$

that is, the structure maps of each vector bundle structure are vector bundle morphisms relative to the other structure [22].

Dualizing TE over E , we get the double vector bundle

$$\begin{array}{ccc} T^*E & \xrightarrow{c_E} & E \\ r_E \downarrow & & \downarrow q_E \\ E^* & \xrightarrow{q_{E^*}} & M \end{array} .$$

The map r_E is given as follows. For α_{e_m} , $r_E(\alpha_{e_m}) \in E_m^*$,

$$\langle r_E(\alpha_{e_m}), f_m \rangle = \langle \alpha_{e_m}, \left. \frac{d}{dt} \right|_{t=0} e_m + t f_m \rangle$$

for all $f_m \in E_m$. The addition in $T^*E \rightarrow E^*$ is defined as follows. If α_{e_m} and $\beta_{e'_m}$ are such that $r_E(\alpha_{e_m}) = r_E(\beta_{e'_m}) = \xi_m \in E_m^*$, then the sum $\alpha_{e_m} +_{r_E} \beta_{e'_m} \in T_{e_m + e'_m}^* E$ is given by

$$\langle \alpha_{e_m} +_{r_E} \beta_{e'_m}, x_{e_m} +_{Tq_E} x_{e'_m} \rangle = \langle \alpha_{e_m}, x_{e_m} \rangle + \langle \beta_{e'_m}, x_{e'_m} \rangle.$$

Note that $T_{e_m}^* E$ is generated by $\mathbf{d}_{e_m} \ell_\xi$ and $\mathbf{d}_{e_m}(q_E^* \varphi)$ for $\xi \in \Gamma(E^*)$ and $\varphi \in C^\infty(M)$. We have $r_E(\mathbf{d}_{e_m} \ell_\xi) = \xi(m)$ and $r_E(\mathbf{d}_{e_m}(q_E^* \varphi)) = 0_m^{E^*}$. The sum $\mathbf{d}_{e_m} \ell_\xi + r_E \mathbf{d}_{e'_m} \ell_\xi$ equals $\mathbf{d}_{e_m + e'_m} \ell_\xi$.

2.3. Basic facts about connections. In this paper, connections will not be defined on Lie algebroids, but more generally on *dull algebroids*. We make the following definition.

Definition 2.2. A **dull algebroid** is a vector bundle $Q \rightarrow M$ endowed with an anchor, i.e. a vector bundle morphism $\rho_Q : Q \rightarrow TM$ over the identity on M and a bracket $[\cdot, \cdot]_Q$ on $\Gamma(Q)$ with

$$(2.4) \quad \rho_Q[q, q']_Q = [\rho_Q(q), \rho_Q(q')]$$

for all $q, q' \in \Gamma(Q)$, and satisfying the Leibniz identity

$$[\varphi_1 q_1, \varphi_2 q_2]_Q = \varphi_1 \varphi_2 [q_1, q_2]_Q + \varphi_1 \rho_Q(q_1)(\varphi_2) q_2 - \varphi_2 \rho_Q(q_2)(\varphi_1) q_1$$

for all $\varphi_1, \varphi_2 \in C^\infty(M)$, $q_1, q_2 \in \Gamma(Q)$.

In other words, a dull algebroid is a **Lie algebroid** if its bracket is in addition skew-symmetric and satisfies the Jacobi-identity.

Let $(Q \rightarrow M, \rho_Q, [\cdot, \cdot]_Q)$ be a dull algebroid and $B \rightarrow M$ a vector bundle. A Q -connection on B is a map

$$\nabla : \Gamma(Q) \times \Gamma(B) \rightarrow \Gamma(B),$$

with the usual properties. By the properties of a dull algebroid, one can still make sense of the curvature R_∇ of the connection, which is an element of $\Gamma(Q^* \otimes Q^* \otimes B^* \otimes B)$.

The dual connection $\nabla^* : \Gamma(Q) \times \Gamma(B^*) \rightarrow \Gamma(B^*)$ is defined by

$$\langle \nabla_q^* \xi, b \rangle = \rho_Q(q) \langle \xi, b \rangle - \langle \xi, \nabla_q b \rangle$$

for all $q \in \Gamma(Q)$, $b \in \Gamma(B)$ and $\xi \in \Gamma(B^*)$.

2.3.1. The Bott connection associated to a subbundle $F \subseteq TM$. Recall the definition of the Bott connection associated to an involutive subbundle of TM : Let $F \subseteq TM$ be a subbundle, then the Lie bracket on vector fields on M induces a map

$$\tilde{\nabla}^F : \Gamma(F) \times \Gamma(TM) \rightarrow \Gamma(TM/F),$$

defined by

$$\tilde{\nabla}_X^F Y = \overline{[X, Y]}.$$

The subbundle F is involutive if and only if

$$\tilde{\nabla}_X^F X' = 0 \quad \text{for all } X, X' \in \Gamma(F).$$

In that case, the map $\tilde{\nabla}^F$ quotients to a flat connection

$$\nabla^F : \Gamma(F) \times \Gamma(TM/F) \rightarrow \Gamma(TM/F),$$

the Bott connection.

2.3.2. The basic connections associated to a connection on a quasi Lie algebroid. Consider here a dull algebroid $(Q, \rho_Q, [\cdot, \cdot]_Q)$ together with a connection $\nabla : \mathfrak{X}(M) \times \Gamma(Q) \rightarrow \Gamma(Q)$.

Then there are Q -connections on Q and TM , called the **basic connections** and defined as follows [8, 1].

$$\nabla^{\text{bas}} = \nabla^{\text{bas}, Q} : \Gamma(Q) \times \Gamma(Q) \rightarrow \Gamma(Q),$$

$$\nabla_q^{\text{bas}} q' = [q, q'] + \nabla_{\rho_Q(q')} q$$

and

$$\nabla^{\text{bas}} = \nabla^{\text{bas}, TM} : \Gamma(Q) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M),$$

$$\nabla_q^{\text{bas}} X = [\rho_Q(q), X] + \rho_Q(\nabla_X q).$$

The **basic curvature** is the map

$$R_{\nabla}^{\text{bas}} : \Gamma(Q) \times \Gamma(Q) \times \mathfrak{X}(M) \rightarrow \Gamma(Q),$$

$$R_{\nabla}^{\text{bas}}(q, q')(X) = -\nabla_X[q, q'] + [\nabla_X q, q'] + [q, \nabla_X q'] + \nabla_{\nabla_{q'}^{\text{bas}} X} q - \nabla_{\nabla_q^{\text{bas}} X} q'.$$

The basic curvature is tensorial and we have the identities

$$\nabla^{\text{bas}, TM} \circ \rho_Q = \rho_Q \circ \nabla^{\text{bas}, Q}, \quad \rho_Q \circ R_{\nabla}^{\text{bas}} = R_{\nabla^{\text{bas}, TM}} \quad \text{and} \quad R_{\nabla}^{\text{bas}} \circ \rho_Q = R_{\nabla^{\text{bas}, Q}}.$$

2.3.3. Connections on a vector bundle E , splittings of TE and the Lie bracket on $\mathfrak{X}(E)$. We recall here the relation between a connection on a vector bundle E and the Lie bracket of vector fields on E .

Let $q_E : E \rightarrow M$ be a vector bundle and $\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ a connection. Then for each $e_m \in E$ and $v_m \in TM$, we can define the vector

$$\widetilde{v_m, e_m} = T_m e v_m - \left. \frac{d}{dt} \right|_{t=0} e_m + t \nabla_{v_m} e \in T_{e_m} E$$

for any section $e \in \Gamma(E)$ such that $e(m) = e_m$. We have

$$\widetilde{v_m, e_m}(\ell_\xi) = v_m \langle \xi, e \rangle - \langle \xi(m), \nabla_{v_m} e \rangle = \ell_{\nabla_{v_m}^* \xi}(e_m)$$

and

$$\widetilde{v_m, e_m}(q_E^* \varphi) = v_m(\varphi)$$

for all $\varphi \in C^\infty(M)$ and $\xi \in \Gamma(E^*)$. The set of all vectors in TE defined in this manner is a subbundle H_∇ of $p_E : TE \rightarrow E$ that is in direct sum with the vertical space $V := T^{q_E} E = \{v_{e_m} \in TE \mid T_{e_m} q_E v_{e_m} = 0\}$:

$$TE \cong V \oplus H_\nabla \rightarrow E.$$

For each section $X \in \mathfrak{X}(M)$, the vector field $\tilde{X} \in \Gamma(H_\nabla) \subseteq \mathfrak{X}(E)$ is defined by $\tilde{X}(e_m) = \widetilde{X(m), e_m}$. For all functions $\varphi \in C^\infty(M)$ and sections $\xi \in \Gamma(E^*)$, we have

$$(2.5) \quad \tilde{X}(\ell_\xi) = \ell_{\nabla_X^* \xi}, \quad \tilde{X}(q_E^* \varphi) = q_E^*(X(\varphi)), \quad f^\uparrow(\ell_\xi) = q_E^* \langle \xi, e \rangle, \quad f^\uparrow(q_E^* \varphi) = 0.$$

Conversely, consider a splitting $TE \cong V \oplus H$ of $TE \rightarrow E$. Then, since $H \cong TE/V$ is isomorphic to the pullback $q_E^! TM \rightarrow E$, we find for each vector field $X \in \mathfrak{X}(M)$ a unique section \tilde{X} of H such that $\tilde{X} \sim_{q_E} X$. Using this uniqueness, one can show easily that the pair (X, \tilde{X}) defines a vector bundle morphism

$$\begin{array}{ccc} E & \xrightarrow{\tilde{X}} & TE \\ q_E \downarrow & & \downarrow Tq_E \\ M & \xrightarrow{X} & TM \end{array}$$

and that we have the equality

$$\widetilde{\varphi \cdot X} = q_E^* \varphi \cdot \tilde{X}$$

for all $\varphi \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$. Using this and $\ell_{\varphi \cdot \xi} = q_E^* \varphi \cdot \ell_\xi$ for all $\varphi \in C^\infty(M)$ and $\xi \in \Gamma(E^*)$, one can then *define* a connection $\nabla^H : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ by setting

$$\tilde{X}(\ell_\xi) =: \ell_{\nabla^H X^* \xi}$$

for all $X \in \mathfrak{X}(M)$, $\xi \in \Gamma(E^*)$.

This shows the correspondence of the two definitions of a connection; the first as the map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E),$$

the second as a splitting

$$TE \cong V \oplus H \rightarrow E.$$

Given ∇ or H_∇ , it is easy to see using the equalities in (2.5) that

$$\begin{aligned} [\tilde{X}, \tilde{Y}] &= \widetilde{[X, Y]} - R_\nabla(X, Y)^\uparrow, \\ [\tilde{X}, e^\uparrow] &= (\nabla_X e)^\uparrow, \\ [e^\uparrow, f^\uparrow] &= 0 \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$ and $e, f \in \Gamma(E)$. That is, the Lie bracket of vector fields on E can be described using the connection. The connection itself can also be seen as a suitable quotient of the Bott connection ∇^{H_∇} :

$$\nabla_{\tilde{X}}^{H_\nabla} e^\uparrow = \overline{(\nabla_X e)^\uparrow}$$

for all $e \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$, i.e. the Bott connection associated to H_∇ restricts well to linear (horizontal) and vertical sections.

Remark 2.3. To any derivation $D : \Gamma(E) \rightarrow \Gamma(E)$ over a vector field $X \in \mathfrak{X}(M)$, we can associate a vector field $\hat{D} \in \mathfrak{X}(E)$ as follows:

$$\hat{D}(q_E^* \varphi) = q_E^*(X(\varphi))$$

for all $\varphi \in C^\infty(M)$, and

$$\hat{D}(\ell_\xi) = \ell_{D^* \xi}$$

for all $\xi \in \Gamma(E^*)$. Here, $D^* : \Gamma(E^*) \rightarrow \Gamma(E^*)$ is the dual of the derivation D , i.e.

$$\langle D^*(\xi), e \rangle = X \langle \xi, e \rangle - \langle \xi, D(e) \rangle$$

for all $e \in \Gamma(E)$ and $\xi \in \Gamma(E^*)$.

We will use this notation in the paper. Given a connection $\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$, the vector field \tilde{X} defined as above is just $\widehat{\nabla_X}$.

3. DORFMAN CONNECTIONS: DEFINITION AND EXAMPLES

Definition 3.1. Let $(Q \rightarrow M, \rho_Q, [\cdot, \cdot]_Q)$ be a dull algebroid. Let $B \rightarrow M$ be a vector bundle with a fiberwise pairing $\langle \cdot, \cdot \rangle : Q \times_M B \rightarrow \mathbb{R}$ and a map $\mathbf{d}_B : C^\infty(M) \rightarrow \Gamma(B)$ such that

$$(3.6) \quad \langle q, \mathbf{d}_B \varphi \rangle = \rho_Q(q)(\varphi)$$

for all $q \in \Gamma(Q)$ and $\varphi \in C^\infty(M)$. Then $(B, \mathbf{d}_B, \langle \cdot, \cdot \rangle)$ will be called a **pre-dual** of Q and Q and B are said to be **paired by** $\langle \cdot, \cdot \rangle$.

Remark 3.2. Note that if the pairing is nondegenerate, then $(B \rightarrow M, \mathbf{d}_B, \langle \cdot, \cdot \rangle)$ is the **dual** of $(Q \rightarrow M, \rho_Q, [\cdot, \cdot]_Q)$ and $\mathbf{d}_{Q^*} : C^\infty(M) \rightarrow \Gamma(Q^*)$ is defined by (3.6). We have then $\mathbf{d}_{Q^*} \varphi = \rho_Q^* \mathbf{d}_B \varphi$, i.e. $\langle \mathbf{d}_{Q^*} \varphi, q \rangle = \rho_Q(q)(\varphi)$ for all $q \in \Gamma(Q)$ and $\varphi \in C^\infty(M)$.

The main definition of this paper is the following.

Definition 3.3. Let $(Q \rightarrow M, \rho_Q, [\cdot, \cdot]_Q)$ be a dull algebroid and $(B \rightarrow M, \mathbf{d}_B, \langle \cdot, \cdot \rangle)$ a pre-dual of Q .

(1) A **Dorfman (Q-)connection on B** is an \mathbb{R} -bilinear map

$$\Delta : \Gamma(Q) \times \Gamma(B) \rightarrow \Gamma(B)$$

such that

- (a) $\Delta_{\varphi q} b = \varphi \Delta_q b + \langle q, b \rangle \cdot \mathbf{d}_B \varphi$,
 - (b) $\Delta_q(\varphi b) = \varphi \Delta_q b + \rho_Q(q)(\varphi) b$,
 - (c) $\rho(q) \langle q', b \rangle = \langle [q, q']_Q, b \rangle + \langle q', \Delta_q b \rangle$
- for all $\varphi \in C^\infty(M)$, $q, q' \in \Gamma(Q)$, $b \in \Gamma(B)$.

(2) The curvature of Δ is the map

$$R_\Delta : \Gamma(Q) \times \Gamma(Q) \rightarrow \Gamma(B^* \otimes B),$$

defined on $q, q' \in \Gamma(Q)$ by $R_\Delta(q, q') := \Delta_q \Delta_{q'} - \Delta_{q'} \Delta_q - \Delta_{[q, q']}_Q$.

The failure of a Dorfman connection to be a connection is hence measured by the map \mathbf{d}_B and the pairing of Q with B . Before we go on with examples, we have to check that $R_\Delta(q, q')$ is an element of $\Gamma(B^* \otimes B)$ for all $q, q' \in \Gamma(Q)$. But this is a straightforward computation, and we omit the proof of the following proposition.

Proposition 3.4. *Let $(Q \rightarrow M, \rho_Q, [\cdot, \cdot]_Q)$ be a dull algebroid and $(B, \mathbf{d}_B, \langle \cdot, \cdot \rangle)$ a pre-dual of Q . Let Δ be a Dorfman Q -connection on B . Then:*

(1) *For all $\varphi \in C^\infty(M)$ and $q, q' \in \Gamma(Q)$, $b \in \Gamma(B)$, we have*

$$R_\Delta(q, q')(\varphi \cdot b) = \varphi \cdot R_\Delta(q, q').$$

(2) *For all $q_1, q_2, q_3 \in \Gamma(Q)$ and $b \in \Gamma(B)$, we have*

$$\langle R_\Delta(q_1, q_2)(b), q_3 \rangle = \langle [[q_1, q_2]_Q, q_3]_Q + [q_2, [q_1, q_3]_Q]_Q - [q_1, [q_2, q_3]_Q]_Q, b \rangle.$$

In particular, if Q is a Lie algebroid, the Dorfman connection is always flat in this sense.

Remark 3.5. Note that this doesn't mean that the curvature of a Dorfman connection vanishes everywhere if Q is a Lie algebroid, since the pairing of Q and B can be degenerate. We will see a trivial example for this phenomenon in Example 3.6.

Example 3.6. Let $(Q \rightarrow M, \rho_Q, [\cdot, \cdot]_Q)$ be a dull algebroid and $B \rightarrow M$ a vector bundle. Take the pairing $\langle \cdot, \cdot \rangle : Q \times_M B \rightarrow \mathbb{R}$ and the map $\mathbf{d}_B : C^\infty(M) \rightarrow \Gamma(B)$ to be trivial. Then any Q -connection on B is also a Dorfman connection.

Example 3.7. The easiest non-trivial example of a Dorfman connection is the map

$$\mathcal{L} : \Gamma(Q) \times \Gamma(Q^*) \rightarrow \Gamma(Q^*),$$

$$\langle \mathcal{L}_q \xi, q' \rangle = \rho_Q(q) \langle q, \xi \rangle - \langle \xi, [q, q']_Q \rangle,$$

for a dull algebroid $(Q \rightarrow M, \rho_Q, [\cdot, \cdot]_Q)$ and its dual (Q^*, \mathbf{d}_{Q^*}) , i.e. with the canonical pairing $Q \times_M Q^* \rightarrow \mathbb{R}$ and $\mathbf{d}_{Q^*} = \rho_Q^* \mathbf{d} : C^\infty(M) \rightarrow \Gamma(Q^*)$.

The third property of a Dorfman connection is immediate by definition of \mathcal{L} and the first two properties are easily verified. The curvature vanishes if and only if $(Q, \rho_Q, [\cdot, \cdot]_Q)$ is a Lie algebroid.

Let $(E \rightarrow M, \rho : E \rightarrow TM, \langle \cdot, \cdot \rangle, [\cdot, \cdot])$ be a Courant algebroid. If K is a subalgebroid of E , the (in general singular) distribution $S := \rho(K) \subseteq TM$ is algebraically involutive and we can define the “singular” Bott connection

$$\nabla^S : \Gamma(S) \times \frac{\mathfrak{X}(M)}{\Gamma(S)} \rightarrow \frac{\mathfrak{X}(M)}{\Gamma(S)}$$

by

$$\nabla_s^S \bar{X} = \overline{[s, X]}$$

for all $X \in \mathfrak{X}(M)$ and $s \in \Gamma(S)$.

The anchor $\rho : E \rightarrow TM$ induces a map $\bar{\rho} : \Gamma(E/K) \rightarrow \mathfrak{X}(M)/\Gamma(S)$, $\bar{\rho}(\bar{e}) = \rho(e) + \Gamma(S)$.

Proposition 3.8. *Let $E \rightarrow M$ be a Courant algebroid and $K \subseteq E$ an isotropic subalgebroid. Then the map*

$$\Delta : \Gamma(K) \times \Gamma(E/K) \rightarrow \Gamma(E/K)$$

$$\Delta_k \bar{e} = \overline{[k, e]}$$

is a Dorfman connection. The dull algebroid structure on K is its induced Lie algebroid structure, the map $\mathbf{d}_{E/K}$ is just $\mathcal{D} + \Gamma(K)$ and the pairing $\langle \cdot, \cdot \rangle : K \times_M (E/K) \rightarrow \mathbb{R}$ is the natural pairing induced by the pairing on E .

We have

$$\bar{\rho}(\Delta_k \bar{e}) = \nabla_{\rho(k)}^S \bar{\rho}(\bar{e})$$

for all $k \in \Gamma(K)$ and $\bar{e} \in \Gamma(E/K)$.

Remark 3.9. (1) Because of the analogy of the Dorfman connection in the last proposition with the Bott connection defined by involutive subbundles of TM , we name this Dorfman connection the *Bott–Dorfman connection associated to K* .

(2) Note that if K is a Dirac structure D in E , then $E/D \simeq D^*$ and the Dorfman connection is just the Lie algebroid derivative of D on $\Gamma(D^*)$.

3.1. Example: reduction of Courant algebroids. Let $E \rightarrow M$ be a Courant algebroid and $K \subseteq E$ an isotropic subalgebroid. Choose $k, k' \in \Gamma(K)$ and $e \in \Gamma(K^\perp)$. Then the equality

$$\langle [k, e], k' \rangle = -\langle e, [k, k'] \rangle + \rho(k) \langle e, k' \rangle = 0$$

shows that $[k, e] \in \Gamma(K^\perp)$. Thus, the Dorfman connection in Proposition 3.8 restricts to a flat connection

$$\nabla : \Gamma(K) \times \Gamma(K^\perp/K) \rightarrow \Gamma(K^\perp/K).$$

Assume that $\rho(K) \subseteq TM$ is simple, i.e. it has constant rank, is hence Frobenius integrable and its space of leaves is a smooth manifold such that the canonical projection is a smooth surjective submersion. The ∇ -parallel sections of K^\perp/K are the sections that project to the quotient $(K^\perp/K)/\nabla \rightarrow M/\rho(K)$ and the properties of ∇ :

- (1) ∇ is flat
- (2) ρ intertwines ∇ with the Bott connection $\nabla^{\rho(K)}$,

can be used to show as in [29] that, under necessary regularity conditions, the quotient has the structure of a Courant algebroid over $M/\rho(K)$. (The ∇ -parallel sections of K^\perp/K are exactly the sections that are called basic in [29].)

Example 3.10. Consider the standard Courant algebroid $TM \oplus T^*M \rightarrow M$ over a smooth manifold M and an involutive subbundle $F \subseteq TM$. The Dorfman connection associated to $F \oplus \{0\} \subseteq TM \oplus T^*M$ is given by

$$\Delta_X(\overline{Y, \beta}) = \overline{(\mathcal{L}_X Y, \mathcal{L}_X \beta)}$$

for all $X \in \Gamma(F)$ and $(Y, \beta) \in \mathfrak{X}(M) \times \Omega^1(M)$. Hence, from the preceding example, we recover the fact that the sections of $TM \oplus T^*M$ that project to the reduced Courant algebroid $T(M/F) \oplus T^*(M/F) \rightarrow (M/F)$ are the sections (Y, β) of $TM \oplus F^\circ$ that are invariant under F , i.e. such that $\mathcal{L}_X(Y, \beta) \in \Gamma(F \oplus \{0\})$ for all $X \in \Gamma(F)$ (e.g. [17]).

Note that $(TM \oplus F^\circ)/(F \oplus 0) \simeq (TM/F) \oplus F^\circ$ and $(TM/F)^* \simeq F^\circ$. The connection

$$\nabla : \Gamma(F) \times \Gamma(TM/F \oplus F^\circ) \rightarrow \Gamma(TM/F \oplus F^\circ)$$

is, modulo these identifications, just the product of the Bott connection ∇^F and its dual

$$\nabla^{F^*} : \Gamma(F) \times \Gamma((TM/F)^*) \rightarrow \Gamma((TM/F)^*).$$

3.2. Generalized complex structures and Dorfman connections. Let V be a vector space. Consider a linear endomorphism \mathcal{J} of $V \oplus V^*$ such that $\mathcal{J}^2 = -\text{Id}_V$ and \mathcal{J} is orthogonal with respect to the inner product

$$(X + \xi, Y + \eta)_V = \xi(Y) + \eta(X), \quad \forall X, Y \in V, \xi, \eta \in V^*.$$

Such a linear map is called a *linear generalized complex structure* by Hitchin [13]. The complexified vector space $(V \oplus V^*) \otimes \mathbb{C}$ decomposes as the direct sum

$$(V \oplus V^*) \otimes \mathbb{C} = E_+ \oplus E_-$$

of the eigenbundles of \mathcal{J} corresponding to the eigenvalues $\pm i$ respectively, i.e.,

$$E_{\pm} = \{(X + \xi) \mp i\mathcal{J}(X + \xi) \mid X + \xi \in V \oplus V^*\}.$$

Both eigenspaces are maximal isotropic with respect to $\langle \cdot, \cdot \rangle$ and they are complex conjugate to each other.

The linear generalized complex structures are in 1-1 correspondence with the splittings $(V \oplus V^*) \otimes \mathbb{C} = E_+ \oplus E_-$ with E_{\pm} maximal isotropic and $E_- = \overline{E_+}$.

Now, let M be a manifold and \mathcal{J} a bundle endomorphism of $TM \oplus T^*M$ such that $\mathcal{J}^2 = -\text{Id}_{TM \oplus T^*M}$, and \mathcal{J} is orthogonal with respect to $\langle \cdot, \cdot \rangle_M$. Then \mathcal{J} is a *generalized almost complex structure*. In the associated eigenbundle decomposition

$$T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M = E_+ \oplus E_-,$$

if $\Gamma(E_+)$ is closed under the (complexified) Courant bracket, then E_+ is a (complex) Dirac structure on M and one says that \mathcal{J} is a *generalized complex structure* [13, 12]. In this case, E_- must also be a Dirac structure since $E_- = \overline{E_+}$. Indeed (E_+, E_-) is a complex Lie bialgebroid in the sense of Mackenzie-Xu [23], in which E_+ and E_- are complex conjugate to each other.

Since $E_- = \overline{E_+}$ and $E_- \cap E_+ = \{0\}$, we have a vector bundle isomorphism

$$\frac{T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M}{E_-} \rightarrow E_+$$

that is given by²

$$\hat{d} \mapsto \frac{1}{2}(d - i\mathcal{J}(d)).$$

The Dorfman connection

$$\Delta^{E_-} : \Gamma(E_-) \times \Gamma(E_+) \rightarrow \Gamma(E_+)$$

is then simply given by

$$\Delta_{d_-}^{E_-} d_+ = \frac{1}{2}([d_-, d_+] - i\mathcal{J}[d_-, d_+]).$$

In the same manner,

$$\Delta^{E_+} : \Gamma(E_+) \times \Gamma(E_-) \rightarrow \Gamma(E_-)$$

is given by

$$\Delta_{d_+}^{E_+} d_- = \frac{1}{2}([d_+, d_-] + i\mathcal{J}[d_+, d_-])$$

for all $d_- \in \Gamma(E_-)$ and $d_+ \in \Gamma(E_+)$.

It would be interesting to study in more detail the properties of these two (flat!) Dorfman connections, maybe in the spirit of the results in [19].

4. SPLITTINGS OF $TE \oplus T^*E$

Consider a vector bundle $q_E : E \rightarrow M$. In this section, the vector bundle $TM \oplus E^*$ will always be anchored by the projection $\text{pr}_{TM} : TM \oplus E^* \rightarrow TM$ and the dual $E \oplus T^*M$ will always be paired with $TM \oplus E^*$ by the canonical pairing. The map $\mathbf{d}_{E \oplus T^*M} : C^\infty(M) \rightarrow \Gamma(E \oplus T^*M)$ will consequently always be given by

$$\mathbf{d}_{E \oplus T^*M} = \text{pr}_{TM}^* \circ \mathbf{d},$$

i.e.

$$\mathbf{d}_{E \oplus T^*M} \varphi = (0, \mathbf{d}\varphi)$$

for all $\varphi \in C^\infty(M)$. A Dorfman connection Δ will here always be a $TM \oplus E^*$ -Dorfman connection on $E \oplus T^*M$, with dual $[[\cdot, \cdot]]_\Delta$. Note that since the pairing is nondegenerate, the

²To avoid confusions, we write in this subsection \hat{d} for the class of $d \in \Gamma(T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M)$ in $(T_{\mathbb{C}}M \oplus T_{\mathbb{C}}^*M)/E_-$.

Dorfman connection is completely determined by its dual structure, the associated bracket $\llbracket \cdot, \cdot \rrbracket_\Delta$ and vice-versa. Hence, we can say here that a Dorfman connection is equivalent to a dull algebroid $(TM \oplus E^*, \text{pr}_{TM}, \llbracket \cdot, \cdot \rrbracket_\Delta)$. It is easy to see, using Proposition 3.4, that the curvature R_Δ will always vanish on $(TM \oplus E^*) \otimes (TM \oplus E^*) \otimes (0 \oplus T^*M)$.

Recall from §2.3.3 that an ordinary connection

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$$

corresponds to a splitting

$$TE = T^{qE}E \oplus H_\nabla \rightarrow E.$$

We show in this section that a Dorfman connection

$$\Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \rightarrow \Gamma(E \oplus T^*M)$$

is the same as a splitting

$$TE \oplus T^*E = (V \oplus V^\circ) \oplus L_\Delta$$

where, as before, $V := T^{qE}E$. The subbundle L_Δ is an almost Dirac structure on E if and only if the bracket $\llbracket \cdot, \cdot \rrbracket_\Delta$ dual to the Dorfman connection is skew-symmetric and the failure of $\Gamma(L_\Delta)$ to be closed under the Dorfman bracket is measured by the curvature R_Δ .

In the following, for any section (f, θ) of $E \oplus T^*M$, the section $(f, \theta)^\uparrow$ of $V \oplus V^\circ$ is the pair defined by

$$(f, \theta)^\uparrow(e_m) = \left(\frac{d}{dt} \Big|_{t=0} e_m + tf(m), (T_{e_m}qE)^*\theta(m) \right)$$

for all $e_m \in E$. Note that the pairs $(f, \theta)^\uparrow(e_m)$ for all $(f, \theta) \in \Gamma(E \oplus T^*M)$, span by construction the fiber $(V \oplus V^\circ)(e_m)$.

4.1. The standard almost Dorfman connection associated to an usual connection on E . We start in this subsection with a simple, motivating example.

Definition 4.1. Let $E \rightarrow M$ be a vector bundle with a connection $\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$. Then the standard Dorfman connection associated to ∇ is the map

$$\begin{aligned} \Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) &\rightarrow \Gamma(E \oplus T^*M), \\ \Delta_{(X, \xi)}(e, \theta) &= (\nabla_X e, \mathcal{L}_X \theta + \langle \nabla^* \xi, e \rangle). \end{aligned}$$

The dual bracket is in this case defined by

$$\llbracket (X, \xi), (Y, \eta) \rrbracket_\Delta = ([X, Y], \nabla_X^* \eta - \nabla_Y^* \xi)$$

for all $(X, \xi), (Y, \eta) \in \Gamma(TM \oplus E^*)$.

Proposition 4.2. Let $E \rightarrow M$ be a vector bundle endowed with a connection ∇ .

(1) The curvature of the standard Dorfman connection Δ associated to ∇ is given by

$$R_\Delta((X, \xi), (Y, \eta)) = (R_\nabla(X, Y), R_{\nabla^*}(\cdot, X)(\eta) - R_{\nabla^*}(\cdot, Y)(\xi)).$$

(2) $(TM \oplus E^*, \text{pr}_{TM}, \llbracket \cdot, \cdot \rrbracket_\Delta)$ is a Lie algebroid if and only if ∇ is flat.

Proof. The first claim is proved by a straightforward computation. The second statement follows then, using Proposition 3.4 and the fact that the pairing is here nondegenerate. \square

Proposition 4.3. Let $E \rightarrow M$ be a vector bundle endowed with a connection ∇ . and let Δ be the standard Dorfman connection associated to ∇ . For any section $(X, \xi) \in \Gamma(TM \oplus E^*)$, set $\widetilde{(X, \xi)} \in \mathfrak{X}(E) \times \Omega^1(E)$,

$$\begin{aligned} \widetilde{(X, \xi)}(e_m) &= (T_m eX(m), \mathbf{d}_{e_m} \ell_\xi) - (\Delta_{(X, \xi)}(e, 0))^\uparrow(e_m) \\ &= (T_m eX(m), \mathbf{d}_{e_m} \ell_\xi) - \left(\frac{d}{dt} \Big|_{t=0} e_m + t \nabla_X e, (T_{e_m} qE)^* \langle \nabla^* \xi, e \rangle \right). \end{aligned}$$

The subbundle L_Δ spanned by these sections is equal to $H_\nabla \oplus H_\nabla^\circ$. Hence, the standard Dorfman connection associated to a connection ∇ is the same as a splitting

$$TE \oplus T^*E \cong (V \oplus V^\circ) \oplus (H_\nabla \oplus H_\nabla^\circ),$$

the sum of a Dirac structure and an almost Dirac structure.

Proof. We just have to check that the space spanned by the sections $\widetilde{(X, \xi)}$ is equal to $H_\nabla \oplus H_\nabla^\circ$. But since $\Delta_{(X,0)}(e, 0) = (\nabla_X e, 0)$ and $\Delta_{(0,\xi)}(e, 0) = (0, \langle \nabla^* \xi, e \rangle)$, the subbundle that we are considering is spanned by the sections $\widetilde{(X, 0)} = (\tilde{X}, 0)$ and $\widetilde{(0, \xi)} = (0, \tilde{\xi})$, i.e. it is the direct sum of a subbundle of TE and a subbundle of T^*E . The tangent part is obviously equal to H_∇ and the cotangent part is easily seen to be H_∇° . \square

4.2. Almost Dorfman connection associated to a splitting $TE \oplus T^*E \cong (V \oplus V^\circ) \oplus L$. Consider now a splitting

$$TE \oplus T^*E \cong (V \oplus V^\circ) \oplus L \rightarrow E,$$

where, as before, $V := T^{q_E}E$. Recall that the vector bundle morphism

$$\Phi_E := (q_{E*}, r_E) : \begin{array}{ccc} TE \oplus T^*E & \longrightarrow & TM \oplus E^* \\ \downarrow & & \downarrow \\ E & \xrightarrow{q_E} & M \end{array}$$

is a fibration of vector bundles over the projection $q_E : E \rightarrow M$.

That is,

$$\begin{array}{ccc} TE \oplus T^*E & \xrightarrow{q_E^! \Phi_E} & q_E^!(TM \oplus E^*) \\ \downarrow & & \downarrow \\ E & \xrightarrow{\quad} & E \end{array}$$

is a surjective vector bundle morphism over the identity on E . Since $V \oplus V^\circ$ is the kernel of Φ_E , the diagram above factors as

$$\begin{array}{ccc} L \simeq (TE \oplus T^*E)/(V \oplus V^\circ) & \xrightarrow{q_E^! \Phi_E} & q_E^!(TM \oplus E^*) \\ \downarrow & & \downarrow \\ E & \xrightarrow{\quad} & E \end{array}$$

and we find that for any section (X, ξ) of $TM \oplus E^*$, there exists a unique section $\widetilde{(X, \xi)}$ of L such that

$$\Phi_E \circ \widetilde{(X, \xi)} = (X, \xi) \circ q_E.$$

Note that by the uniqueness of the section $\widetilde{(X, \xi)}$ of L over (X, ξ) , we have $\varphi \cdot \widetilde{(X, \xi)} = q_E^* \varphi \cdot \widetilde{(X, \xi)}$ for all $\varphi \in C^\infty(M)$.

We start by proving the following observation.

Lemma 4.4. *Choose $(X, \xi), (Y, \eta) \in \Gamma(TM \oplus E^*)$. Then*

$$e_m \mapsto \langle \widetilde{(X, \xi)}(e_m), (T_m e Y(m), \mathbf{d}_{e_m} \ell_\eta) \rangle - Y(m) \langle \xi, e \rangle,$$

where $e \in \Gamma(E)$ is such that $e(m) = e_m$, defines a linear map on E .

Proof. Choose first $e, f \in \Gamma(E)$. Then we have

$$\Phi_E(\widetilde{(X, \xi)(e(m))} +_{\Phi_E} \widetilde{(X, \xi)(f(m))}) = (X, \xi)(m)$$

and

$$\Phi_E(\widetilde{(X, \xi)(e(m) + f(m))}) = (X, \xi)(m).$$

Since $\widetilde{(X, \xi)(e(m))} +_{\Phi_E} \widetilde{(X, \xi)(f(m))}$ and $\widetilde{(X, \xi)(e(m) + f(m))}$ are both elements of $L_{e_m + f_m}$, we find by definition of (X, ξ) that

$$\widetilde{(X, \xi)(e(m))} +_{\Phi_E} \widetilde{(X, \xi)(f(m))} = \widetilde{(X, \xi)(e(m) + f(m))}.$$

By definition of the addition in the Φ_E -fibers, we find also

$$(T_m e Y(m), \mathbf{d}_{e_m} \ell_\eta) +_{\Phi_E} (T_m f Y(m), \mathbf{d}_{f_m} \ell_\eta) = (T_m(e + f)Y(m), \mathbf{d}_{e_m + f_m} \ell_\eta)$$

if $e, f \in \Gamma(E)$ are such that $e(m) = e_m$ and $f(m) = f_m$. Again by definition of the addition in the Φ_E -fibers, we get hence

$$\begin{aligned} & \langle \widetilde{(X, \xi)(e(m) + f(m))}, (T_m(e + f)Y(m), \mathbf{d}_{e_m + f_m} \ell_\eta) \rangle \\ &= \langle \widetilde{(X, \xi)(e(m))}, (T_m e Y(m), \mathbf{d}_{e_m} \ell_\eta) \rangle + \langle \widetilde{(X, \xi)(f(m))}, (T_m f Y(m), \mathbf{d}_{f_m} \ell_\eta) \rangle. \end{aligned}$$

In particular, we find

$$\langle \widetilde{(X, \xi)(r \cdot e(m))}, (T_m(r \cdot e)Y(m), \mathbf{d}_{r \cdot e_m} \ell_\eta) \rangle = r \cdot \langle \widetilde{(X, \xi)(e(m))}, (T_m e Y(m), \mathbf{d}_{e_m} \ell_\eta) \rangle$$

for all $r \in \mathbb{N}$. The same equality follows then for $r \in \mathbb{Q}$ and by continuity for all $r \in \mathbb{R}$.

Choose $\varphi \in C^\infty(M)$ and set $\varphi(m) = \alpha$. Then

$$\begin{aligned} & \langle \widetilde{(X, \xi)(\alpha e_m)}, (T_m(\varphi \cdot e)Y(m), \mathbf{d}_{\alpha e_m} \ell_\eta) \rangle - Y(m) \langle \xi, \varphi \cdot e \rangle \\ &= \langle \widetilde{(X, \xi)(\alpha e_m)}, (T_m(\alpha e)Y(m) + Y(m)(\varphi) e^\uparrow(\alpha e_m), \mathbf{d}_{\alpha e_m} \ell_\eta) \rangle - Y(m) \langle \xi, \varphi \cdot e \rangle \\ &= \langle \widetilde{(X, \xi)(\alpha e_m)}, (T_m(\alpha e)Y(m), \mathbf{d}_{\alpha e_m} \ell_\eta) \rangle - \alpha Y(m) \langle \xi, \cdot e \rangle \\ &= \alpha \cdot (\langle \widetilde{(X, \xi)(e_m)}, (T_m e Y(m), \mathbf{d}_{e_m} \ell_\eta) \rangle - Y(m) \langle \xi, \cdot e \rangle). \end{aligned}$$

Thus, we have shown that the function

$$e_m \mapsto \langle \widetilde{(X, \xi)(e_m)}, (T_m e Y(m), \mathbf{d}_{e_m} \ell_\eta) \rangle - Y(m) \langle \xi, e \rangle$$

is well-defined, i.e. doesn't depend on the choice of the section e , and linear. \square

Thus, we can consider the map

$$\llbracket \cdot, \cdot \rrbracket_L : \Gamma(TM \oplus E^*) \times \Gamma(TM \oplus E^*) \rightarrow \Gamma(TM \oplus E^*)$$

defined by

$$\langle \llbracket (X, \xi), (Y, \eta) \rrbracket_L, (e_m, 0) \rangle = \langle \widetilde{(X, \xi)(e_m)}, (T_m e Y(m), \mathbf{d}_{e_m} \ell_\eta) \rangle - Y(m) \langle \xi, e \rangle,$$

for any section $e \in \Gamma(E)$ such that $e(m) = e_m$, and

$$\text{pr}_{TM} \llbracket (X, \xi), (Y, \eta) \rrbracket_L = [X, Y].$$

Theorem 4.5. *Let $E \rightarrow M$ be a vector bundle and consider a splitting*

$$TE \oplus T^*E \cong (V \oplus V^\circ) \oplus L \rightarrow E.$$

The triple $(TM \oplus E^, \text{pr}_{TM}, \llbracket \cdot, \cdot \rrbracket_L)$, where $\llbracket \cdot, \cdot \rrbracket_L$ is defined as above, is a dull algebroid.*

Proof. We compute for $e \in \Gamma(E)$, setting $\alpha := \varphi(m)$:

$$\begin{aligned} \langle \llbracket (X, \xi), \varphi \cdot (Y, \eta) \rrbracket_L, (e, 0) \rangle &= \langle \widetilde{(X, \xi)}(e_m), (T_m e(\alpha Y(m)), \mathbf{d}_{e_m}(q_E^* \varphi \cdot \ell_\eta)) \rangle - \alpha Y(m) \langle \xi, e \rangle \\ &= \alpha \langle \widetilde{(X, \xi)}(e_m), (T_m e Y(m), \mathbf{d}_{e_m} \ell_\eta) \rangle - \alpha Y(m) \langle \xi, e \rangle + X(\varphi) \langle \eta, e \rangle \\ &= \langle \varphi \llbracket (X, \xi), (Y, \eta) \rrbracket_L + X(\varphi)(Y, \eta), (e, 0) \rangle(m), \\ \langle \llbracket \varphi \cdot (X, \xi), (Y, \eta) \rrbracket_L, (e, 0) \rangle &= \langle (q_E^* \varphi \widetilde{(X, \xi)})(e_m), (T_m e Y(m), \mathbf{d}_{e_m} \ell_\eta) \rangle \\ &\quad - Y(m)(\varphi) \langle \xi, e \rangle(m) - \varphi(m) Y(m) \langle \xi, e \rangle \\ &= \langle \varphi \llbracket (X, \xi), (Y, \eta) \rrbracket_L - Y(\varphi)(X, \xi), (e, 0) \rangle(m). \end{aligned}$$

Since

$$\langle \llbracket (X, \xi), \varphi \cdot (Y, \eta) \rrbracket_L, (0, \theta) \rangle = \langle \varphi \llbracket (X, \xi), (Y, \eta) \rrbracket_L + X(\varphi)(Y, \eta), (0, \theta) \rangle$$

and

$$\langle \llbracket \varphi \cdot (X, \xi), (Y, \eta) \rrbracket_L, (0, \theta) \rangle = \langle \varphi \llbracket (X, \xi), (Y, \eta) \rrbracket_L - Y(\varphi)(X, \xi), (0, \theta) \rangle$$

hold by construction for all $\theta \in \Omega^1(M)$, we have shown the two Leibniz equalities. Compatibility of pr_{TM} with $\llbracket \cdot, \cdot \rrbracket_L$ is given by construction. \square

Corollary 4.6. *Let $E \rightarrow M$ be a vector bundle and consider a splitting*

$$TE \oplus T^*E \cong (V \oplus V^\circ) \oplus L \rightarrow E.$$

Define the map

$$\Delta^L : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \rightarrow \Gamma(E \oplus T^*M),$$

$$\langle \Delta_{(X, \xi)}^L(e, \theta), (Y, \eta) \rangle = X \langle (e, \theta), (Y, \eta) \rangle - \langle \llbracket (X, \xi), (Y, \eta) \rrbracket_L, (e, \theta) \rangle$$

Then Δ^L is a Dorfman connection.

Remark 4.7. Note that, by definition, we have $\Delta_{(X, \xi)}^L(0, \theta) = (0, \mathcal{L}_X \theta)$ for all $X \in \mathfrak{X}(M)$ and $\theta \in \Omega(M)$. To see this, choose $Y \in \mathfrak{X}(M)$ and compute

$$\langle \Delta_{(X, \xi)}^L(0, \theta), (Y, 0) \rangle = X \langle \theta, Y \rangle - \langle \theta, [X, Y] \rangle.$$

Proof of Corollary 4.6. By construction Δ^L is dual to the dull bracket $\llbracket \cdot, \cdot \rrbracket_L$ on $\Gamma(TM \oplus E^*)$. \square

We end this subsection with a proposition relating directly the Dorfman connection Δ^L with the subbundle $L \subseteq TE \oplus T^*E$.

Proposition 4.8. *Let $E \rightarrow M$ be a vector bundle and consider a splitting*

$$TE \oplus T^*E \cong (V \oplus V^\circ) \oplus L \rightarrow E.$$

Choose $(X, \xi) \in \Gamma(TM \oplus E^*)$. Then the corresponding section of L is given by

$$\widetilde{(X, \xi)}(e_m) = (T_m e X(m), \mathbf{d}_{e_m} \ell_\xi) - \Delta_{(X, \xi)}^L(e, 0)^\uparrow(e_m)$$

for all $e_m \in E$ and $e \in \Gamma(E)$ such that $e(m) = e_m$.

Proof. Since $\Phi_E \widetilde{(X, \xi)}(e_m) = (X, \xi)(m) = \Phi_E(T_m e X(m), \mathbf{d}_{e_m} \ell_\xi)$ for $e_m \in E$, there exists a pair $(f, \theta) \in \Gamma(E \oplus T^*M)$ such that

$$\widetilde{(X, \xi)}(e_m) = (T_m e X(m), \mathbf{d}_{e_m} \ell_\xi) + (f, \theta)^\uparrow(e_m).$$

We compute for $(Y, \eta) \in \Gamma(TM \oplus E^*)$:

$$\begin{aligned}
\langle (f, \theta), (Y, \eta) \rangle(m) &= \langle (f, \theta)^\dagger(e_m), (T_m eY(m), \mathbf{d}_{e_m} \ell_\eta) \rangle \\
&= \langle \widetilde{(X, \xi)}(e_m) - (T_m eX(m), \mathbf{d}_{e_m} \ell_\xi), (T_m eY(m), \mathbf{d}_{e_m} \ell_\eta) \rangle \\
&= \langle \llbracket (X, \xi), (Y, \eta) \rrbracket_L, (e_m, 0) \rangle + Y(m) \langle \xi, e \rangle - X(m) \langle \eta, e \rangle - Y(m) \langle \xi, e \rangle \\
&= \langle \llbracket (X, \xi), (Y, \eta) \rrbracket_L, (e_m, 0) \rangle - X(m) \langle \eta, e \rangle \\
&= -\langle \Delta_{(X, \xi)}^L(e, 0), (Y, \eta) \rangle(m).
\end{aligned}$$

□

4.3. The converse construction. Let $E \rightarrow M$ be a vector bundle and consider a Dorfman connection

$$\Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \rightarrow \Gamma(E \oplus T^*M).$$

We define the subset

$$L_\Delta \subseteq TE \oplus T^*E$$

by

$$L_\Delta(e_m) = \{ (T_m eX(m), \mathbf{d}\ell_\xi(e_m)) - \Delta_{(X, \xi)}(e, 0)^\dagger(e_m) \mid (X, \xi) \in \Gamma(TM \oplus E^*) \}$$

for any section $e \in \Gamma(E)$ such that $e(m) = e_m$.

For any pair $(X, \xi) \in \Gamma(TM \oplus E^*)$, we will write $\widetilde{(X, \xi)} \in \Gamma_E(TE \oplus T^*E) = \mathfrak{X}(E) \times \Omega^1(E)$ for the section defined by

$$\widetilde{(X, \xi)}(e_m) = (T_m eX(m), \mathbf{d}\ell_\xi(e_m)) - \Delta_{(X, \xi)}(e, 0)^\dagger(e_m)$$

for all $e_m \in E$. Note that $\Phi_E \circ \widetilde{(X, \xi)} = (X, \xi) \circ q_E$.

Proposition 4.9. L_Δ is a well-defined subbundle of $TE \oplus T^*E \rightarrow E$ such that $(V \oplus V^\circ) \oplus L_\Delta \xrightarrow{\sim} TE \oplus T^*E$.

Proof. We show that the fiber over e_m doesn't depend on the choice of the section $e \in \Gamma(E)$ such that $e(m) = e_m$. Note first that for any pair $(f, \theta) \in \Gamma(E \oplus T^*M)$, we have

$$\langle (T_m eX(m), \mathbf{d}\ell_\xi(e_m)) - \Delta_{(X, \xi)}(e, 0)^\dagger(e_m), (f, \theta)^\dagger(e_m) \rangle = \langle \theta(m), X(m) \rangle + \langle \xi(m), f(m) \rangle.$$

This pairing doesn't depend on e .

Choose then any connection ∇ on E and a pair $(Y, \eta) \in \Gamma(TM \oplus E^*)$. Consider the pair

$$(Y_\nabla, \eta_\nabla)(e_m) := \left(T_m eY(m) - \frac{d}{dt} \Big|_{t=0} e_m + t \nabla_{Y(m)} e, \mathbf{d}\ell_\eta(e_m) - (T_{e_m} q_E)^* \langle \nabla^* \eta, e \rangle \right).$$

Then

$$\begin{aligned}
&\langle (T_m eX(m), \mathbf{d}\ell_\xi(e_m)) - \Delta_{(X, \xi)}(e, 0)^\dagger(e_m), (Y_\nabla, \eta_\nabla)(e_m) \rangle \\
&= X(m) \langle \eta, e \rangle - \langle \nabla_{X(m)}^* \eta, e \rangle - \langle \eta(m), \text{pr}_E \Delta_{(X, \xi)}(e, 0) \rangle \\
&\quad + Y(m) \langle \xi, e \rangle - \langle \text{pr}_{T^*M} \Delta_{(X, \xi)}(e, 0), Y(m) \rangle + \langle \xi(m), \nabla_{Y(m)} e \rangle \\
&= X(m) \langle \eta, e \rangle - \langle \nabla_{X(m)}^* \eta, e(m) \rangle - \langle (Y, \eta)(m), \Delta_{(X, \xi)}(e, 0) \rangle + \langle \nabla_{Y(m)}^* \xi, e(m) \rangle \\
&= \langle \llbracket (X, \xi), (Y, \eta) \rrbracket_\Delta, (e(m), 0) \rangle - \langle \nabla_{X(m)}^* \eta, e(m) \rangle + \langle \nabla_{Y(m)}^* \xi, e(m) \rangle.
\end{aligned}$$

Again, this does only depend on the value e_m of e at m .

Since pairs (Y_∇, η_∇) and $(f, \theta)^\dagger$ span the whole of $TE \oplus T^*E$ and the pairing is nondegenerate, we have shown that L_Δ is well-defined. The second claim is immediate, using the fact that $\Phi_E \circ \widetilde{(X, \xi)} = (X, \xi)$ for all $(X, \xi) \in \Gamma(TM \oplus E^*)$ and that $V \oplus V^\circ$ is spanned by the sections $(e, \theta)^\dagger$ for $(e, \theta) \in \Gamma(E \oplus T^*M)$. □

The results in the last two subsections are summarized in the following theorem.

Theorem 4.10. *Let $q_E : E \rightarrow M$ be a Lie algebroid. The maps*

$$\begin{aligned}\Delta &\mapsto L_\Delta, \\ \Delta^L &\mapsto L\end{aligned}$$

define a bijection

$$\left\{ \begin{array}{c} (TM \oplus E^*)\text{-Dorfman connections} \\ \Delta \text{ on } E \oplus T^*M \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Splittings} \\ TE \oplus T^*E \cong (V \oplus V^\circ) \oplus L \end{array} \right\}.$$

Since a $(TM \oplus E^*)$ -Dorfman connection Δ on $E \oplus T^*M$ is the same as a dull algebroid structure $(\text{pr}_{TM}, \llbracket \cdot, \cdot \rrbracket_\Delta)$ on $TM \oplus E^*$, we can reformulate this bijection as follows:

$$\left\{ \begin{array}{c} \text{Dull algebroids} \\ (TM \oplus E^*, \text{pr}_{TM}, \llbracket \cdot, \cdot \rrbracket) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Splittings} \\ TE \oplus T^*E \cong (V \oplus V^\circ) \oplus L \end{array} \right\}.$$

4.4. The canonical pairing and the Courant-Dorfman bracket on $TE \oplus T^*E$. We show in this section that the failure of a splitting L of $TE \oplus T^*E$ to be Lagrangian is equivalent to the failure of $\llbracket \cdot, \cdot \rrbracket_L$ to be skew-symmetric, and the failure of its set of sections to be closed under the Courant-Dorfman bracket is measured by the curvature of Δ .

Here and later, we will need the following notation. Let $E \rightarrow M$ be a vector bundle and consider a Dorfman connection $\Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \rightarrow \Gamma(E \oplus T^*M)$. We call $\text{Skew}_\Delta \in \Gamma((TM \oplus E^*) \otimes (TM \oplus E^*) \otimes E^*)$ the tensor defined by

$$\text{Skew}_\Delta(v_1, v_2) = \text{pr}_{E^*}(\llbracket v_1, v_2 \rrbracket_\Delta + \llbracket v_2, v_1 \rrbracket_\Delta)$$

for all $v_1, v_2 \in \Gamma(TM \oplus E^*)$. By the Leibniz identity, this is indeed $C^\infty(M)$ -linear in both arguments. Note that the TM -part of $\llbracket v_1, v_2 \rrbracket_\Delta + \llbracket v_2, v_1 \rrbracket_\Delta$ always vanishes since the Lie bracket of vector fields is skew-symmetric.

Theorem 4.11. *Let $\Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \rightarrow \Gamma(E \oplus T^*M)$ be a Dorfman connection and choose $v, v_1, v_2 \in \Gamma(TM \oplus E^*)$ and $\sigma, \sigma_1, \sigma_2 \in \Gamma(E \oplus T^*M)$. Then*

- (1) $\langle \tilde{v}_1, \tilde{v}_2 \rangle = \ell_{\text{Skew}_\Delta(v_1, v_2)},$
- (2) $\langle \tilde{v}, \sigma^\uparrow \rangle = q_E^* \langle v, \sigma \rangle,$
- (3) $\langle \sigma_1^\uparrow, \sigma_2^\uparrow \rangle = 0.$

Proof. Since the second and third equalities are immediate, we prove only the first one. We write $v_1 = (X, \xi)$, $v_2 = (Y, \eta)$ and compute for any section $e \in \Gamma(E)$:

$$\begin{aligned} & \langle (T_m e X(m), \mathbf{d}\ell_\xi(e_m)) - \Delta_{(X, \xi)}(e, 0)^\uparrow(e_m), (T_m e Y(m), \mathbf{d}\ell_\eta(e_m)) - \Delta_{(Y, \eta)}(e, 0)^\uparrow(e_m) \rangle \\ &= X(m) \langle \eta, e \rangle - \langle \text{pr}_{T^*M} \Delta_{(Y, \eta)}(e, 0), X(m) \rangle - \langle \eta(m), \text{pr}_E \Delta_{(X, \xi)}(e, 0) \rangle \\ & \quad + Y(m) \langle \xi, e \rangle - \langle \text{pr}_{T^*M} \Delta_{(X, \xi)}(e, 0), Y(m) \rangle - \langle \xi(m), \text{pr}_E \Delta_{(Y, \eta)}(e, 0) \rangle \\ &= (X \langle \eta, e \rangle - \langle \Delta_{(Y, \eta)}(e, 0), (X, \xi) \rangle + Y \langle \xi, e \rangle - \langle \Delta_{(X, \xi)}(e, 0), (Y, \eta) \rangle)(m) \\ &= \langle (e, 0), \llbracket v_2, v_1 \rrbracket_\Delta + \llbracket v_1, v_2 \rrbracket_\Delta \rangle. \end{aligned}$$

□

Corollary 4.12. *The bracket $\llbracket \cdot, \cdot \rrbracket_\Delta$ associated to a Dorfman connection Δ is skew-symmetric, if and only if L_Δ is Lagrangian. The corresponding splitting*

$$TE \oplus T^*E \cong (V \oplus V^\circ) \oplus L_\Delta$$

is then the direct sum of the Dirac structure $V \oplus V^\circ$ and the almost Dirac structure L_Δ .

Proof. Since the rank of L_Δ is equal to the dimension of E as a manifold, we have only to show that L_Δ is isotropic if and only if $\llbracket \cdot, \cdot \rrbracket_\Delta$ is skew-symmetric. But this is immediate by the preceding theorem. □

Next, we will see how the Dorfman connection encodes the Courant-Dorfman bracket on linear and core sections. The next theorem shows how integrability of L_Δ is related to the curvature R_Δ of the Dorfman connection.

Theorem 4.13. *Choose v, v_1, v_2 in $\Gamma(TM \oplus E^*)$ and $\sigma, \sigma_1, \sigma_2 \in \Gamma(E \oplus T^*M)$. Then*

- (1) $[\sigma_1^\uparrow, \sigma_2^\uparrow] = 0$,
- (2) $[\widetilde{v}, \sigma^\uparrow] = (\Delta_v \sigma)^\uparrow$,
- (3) $[\widetilde{v}_1, \widetilde{v}_2] = \llbracket v_1, v_2 \rrbracket_\Delta - R_\Delta(v_1, v_2)(\cdot, 0)^\uparrow$.

The proof of this theorem is quite long and technical, it can be found in Appendix C.

Remark 4.14. (1) If the Courant-Dorfman bracket is twisted by a linear closed 3-form H over a map $\bar{H} : TM \wedge TM \rightarrow E^*$ [4], then the bracket $[\widetilde{v}_1, \widetilde{v}_2]$ will be linear over $\llbracket v_1, v_2 \rrbracket_{\bar{H}} = \llbracket v_1, v_2 \rrbracket + (0, \bar{H}(X_1, X_2))$. Note that the Dorfman connection dual to this bracket is $\Delta_v^{\bar{H}} \sigma = \Delta_v \sigma + (0, \langle \bar{H}(X, \cdot), e \rangle)$. A more careful study of general exact Courant algebroids [27] over vector bundles and the corresponding twistings of the Dorfman connections and dull algebroids corresponding to splittings of $TE \oplus T^*E$ will be done later.

- (2) The *Courant bracket*, i.e. the anti-symmetric counterpart of the Courant-Dorfman bracket, is given by

- (a) $[\sigma_1^\uparrow, \sigma_2^\uparrow]_C = 0$,
- (b) $[\widetilde{v}, \sigma^\uparrow]_C = [\widetilde{v}, \sigma^\uparrow] - (0, \frac{1}{2} q_E^* \mathbf{d}\langle v, \sigma \rangle) = (\Delta_v \sigma - (0, \frac{1}{2} \mathbf{d}\langle v, \sigma \rangle))^\uparrow$,
- (c) $[\widetilde{v}_1, \widetilde{v}_2]_C = \llbracket v_1, v_2 \rrbracket_\Delta - R_\Delta(v_1, v_2)(\cdot, 0)^\uparrow - (0, \frac{1}{2} \mathbf{d}\ell_{\text{Skew}_\Delta(v_1, v_2)})$.

Our choice of working with non anti-symmetric Courant algebroids is because of the fact that Dorfman connections describe naturally the Courant-Dorfman bracket: recall Remark 4.7 and Proposition 4.8. Here, we see from (b) that working with a “Courant connection” starting from a splitting like as we did would yield much more complicated formulas and axioms for the connection. This is also why we chose to call the Dorfman connections after I. Dorfman.

The following corollary of Theorem 4.13 is immediate.

Corollary 4.15. *Let $E \rightarrow M$ be a vector bundle and consider a splitting $TE \oplus T^*E = (V \oplus V^\circ) \oplus L$. Then the horizontal space L is a Dirac structure if and only if the corresponding dull algebroid $(TM \oplus E^*, \text{pr}_{TM}, \llbracket \cdot, \cdot \rrbracket_L)$ is a Lie algebroid.*

We will study more general (non-horizontal) Dirac structures on E in the next section. Before that, we end this subsection with some examples.

Example 4.16. Recall that an ordinary connection on a vector bundle $E \rightarrow M$ is a splitting $TE \cong V \oplus H_\nabla$. We have seen in Section 4.1 that the Dorfman connection

$$\begin{aligned} \Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) &\rightarrow \Gamma(E \oplus T^*M), \\ \Delta_{(X, \xi)}(e, \theta) &= (\nabla_X e, \mathcal{L}_X \theta + \langle \nabla^* \xi, e \rangle) \end{aligned}$$

corresponds to the splitting

$$TE \oplus T^*E \cong (V \oplus V^\circ) \oplus (H_\nabla \oplus H_\nabla)^\circ.$$

Example 4.17. Consider a dull algebroid $(A, \rho, [\cdot, \cdot])$ with *skew-symmetric bracket*. We construct a $TM \oplus A$ -connection Δ on $A^* \oplus T^*M$, hence corresponding to a splitting $TA^* \oplus T^*A^* = (V \oplus V^\circ) \oplus L_\Delta$ of the Pontryagin bundle of A^* . Take any connection $\nabla : \mathfrak{X}(M) \times \Gamma(A) \rightarrow \Gamma(A)$ and recall the definition of the basic connection $\nabla^{\text{bas}} : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$ associated to ∇ and the dull algebroid structure on A :

$$\nabla_a^{\text{bas}} b = [a, b] + \nabla_{\rho(b)} a$$

for all $a, b \in \Gamma(A)$. The Dorfman connection

$$\Delta : \Gamma(TM \oplus A) \times \Gamma(A^* \oplus T^*M) \rightarrow \Gamma(A^* \oplus T^*M)$$

is defined by

$$\Delta_{(X,a)}(\xi, \theta) = (\langle \xi, \nabla^{\text{bas}} a \rangle + \nabla_X^* \xi - \rho^* \langle \nabla \cdot a, \xi \rangle, \mathcal{L}_X \theta + \langle \nabla \cdot a, \xi \rangle).$$

The bracket $\llbracket \cdot, \cdot \rrbracket_\Delta$ on sections of $TM \oplus A$ is then given by

$$\llbracket (X, a), (Y, b) \rrbracket_\Delta = ([X, Y], \nabla_X b - \nabla_Y a + \nabla_{\rho(b)} a - \nabla_{\rho(a)} b + [a, b]).$$

Since it is skew-symmetric, the horizontal space L_Δ is in this case Lagrangian. The projection pr_{TM} intertwines obviously this bracket with the Lie bracket of vector fields. The curvature of this Dorfman connection is given by

$$\begin{aligned} \langle R_\Delta((X, a), (Y, b))(\xi, \theta), (Z, c) \rangle &= - \langle \llbracket (X, a), \llbracket (Y, b), (Z, c) \rrbracket_\Delta \rrbracket_\Delta + \text{c.p.}, (\xi, \theta) \rangle \\ (4.7) \quad &= - \langle (R_\nabla(X, Y)c - R_\nabla(\rho(a), Y)c) + \text{c.p.}, \xi \rangle \\ &\quad - \langle (R_\nabla(\rho(a), \rho(b))c - R_\nabla(X, \rho(b))c) + \text{c.p.}, \xi \rangle \\ &\quad - \langle (R_\nabla^{\text{bas}}(a, b)Z - R_\nabla^{\text{bas}}(a, b)\rho(c)) + \text{c.p.}, \xi \rangle \\ &\quad - \langle [a, [b, c]] + [b, [c, a]] + [c, [a, b]], \xi \rangle. \end{aligned}$$

The proof of this formula is a rather long, but straightforward computation that is omitted here.

We will see in the next subsection the signification of this example in terms of the linear almost Poisson structure defined on A^* by the skew-symmetric dull algebroid structure.

Example 4.18. Consider a vector bundle $E \rightarrow M$ endowed with a vector bundle morphism $\sigma : E \rightarrow T^*M$ over the identity and a connection $\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$. Define the Dorfman connection

$$\Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \rightarrow \Gamma(E \oplus T^*M)$$

by

$$\Delta_{(X,\xi)}(e, \theta) = (\nabla_X e, \mathcal{L}_X(\theta - \sigma(e)) + \langle \nabla^*(\sigma^* X + \xi), e \rangle + \sigma(\nabla_X e)).$$

The bracket $\llbracket \cdot, \cdot \rrbracket_\Delta$ on sections of $TM \oplus E^*$ is here given by

$$\llbracket (X, \xi), (Y, \eta) \rrbracket_\Delta = ([X, Y], \nabla_X^*(\eta + \sigma^* Y) - \nabla_Y^*(\xi + \sigma^* X) - \sigma^*[X, Y]).$$

In this case also, L_Δ is Lagrangian.

Here also, we give the curvature of the Dorfman connection in terms of the Jacobiator of the associated bracket:

$$(4.8) \quad \llbracket (X, \xi), \llbracket (Y, \eta), (Z, \gamma) \rrbracket_\Delta \rrbracket_\Delta + \text{c.p.} = (0, R_{\nabla^*}(X, Y)(\gamma + \sigma^* Z) + \text{c.p.}).$$

We will see in the next section how this Dorfman connection is related to the 2-form $\sigma^* \omega_{\text{can}} \in \Omega^2(E)$, where ω_{can} is the canonical symplectic form on T^*M .

4.5. Dirac structures and Dorfman connections. In this subsection, we will consider sub- double vector bundles

$$\begin{array}{ccc} D & \longrightarrow & U \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array} \quad \text{of} \quad \begin{array}{ccc} TE \oplus T^*E & \longrightarrow & TM \oplus E^* \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array}$$

The intersection of such a sub- double vector bundle D with the vertical space $V \oplus V^\circ$ always has constant rank on E and there is a subbundle $K \subseteq E \oplus T^*M$ such that $D \cap (V \oplus V^\circ)$ is spanned over E by the sections k^\dagger for all $k \in \Gamma(K)$. To see that, use for instance [22]. We will call K the **core** of D . The following proposition follows from this observation.

Proposition 4.19. *Let E be a vector bundle endowed with a sub-double vector bundle $D \subseteq TE \oplus T^*E$ over $U \subseteq TM \oplus E^*$ and with core $K \subseteq E \oplus T^*M$. Then there exists a Dorfman connection Δ such that D is spanned by the sections k^\uparrow , for all $k \in \Gamma(K)$ and \tilde{u} for all $u \in \Gamma(U)$.*

The Dorfman connection Δ is then said to be *adapted* to D . Conversely, given a Dorfman connection and two subbundles $U \subseteq TM \oplus E^*$ and $K \subseteq E \oplus T^*M$, we call $D_{U,K,\Delta}$ the sub-double vector bundle that is spanned by σ^\uparrow , for all $\sigma \in \Gamma(K)$ and \tilde{u} for all $u \in \Gamma(U)$.

Definition 4.20. *Two Dorfman connections Δ, Δ' are said to be (U, K) -equivalent if $(\Delta - \Delta')(\Gamma(U) \times \Gamma(E \oplus 0)) \subseteq \Gamma(K)$.*

The following proposition shows that this defines an (U, K) -equivalence relation on the set of Dorfman connections. We will write $[\Delta]_{U,K}$, or simply $[\Delta]$ since there will never be a risk of confusion, for the (U, K) -class of the Dorfman connection Δ . The triple $(U, K, [\Delta])$ will be called a VB-triple in the following. By the next proposition, VB-triples are in one-one correspondence with sub-double vector bundles of $TE \oplus T^*E \rightarrow E$.

Proposition 4.21. *Choose two Dorfman connections Δ, Δ' and assume that Δ is adapted to D . Then Δ' is adapted to D if and only if Δ and Δ' are (U, K) -equivalent.*

Proof. Assume that Δ is adapted to D . Then D is spanned by the sections \tilde{u}^Δ and σ^\uparrow for all $\sigma \in \Gamma(K)$ and $u \in \Gamma(U)$. If Δ and Δ' are (U, K) -equivalent, we have $\tilde{u}^\Delta - \tilde{u}^{\Delta'} = k^\uparrow$ for some $k \in \Gamma(K)$. This implies immediately that Δ' is adapted to D . The converse implication can be proved in a similar manner. \square

The following theorem follows immediately from the results in the preceding subsection.

Theorem 4.22. *Let D be a sub-double vector bundle of $TE \oplus T^*E$ over $U \subseteq TM \oplus E^*$ and $K \subseteq E \oplus T^*M$ and choose a Dorfman connection Δ that is adapted to D . Then*

- (1) D is isotropic if and only if $\text{Skew}_\Delta|_{U \otimes U} = 0$ and $K \subseteq U^\circ$.
- (2) D is Lagrangian if and only if $\text{Skew}_\Delta|_{U \otimes U} = 0$ and $K = U^\circ$.
- (3) $\Gamma(D)$ is closed under the Courant-Dorfman bracket if and only if
 - (a) $\Delta_u k \in \Gamma(K)$ for all $u \in \Gamma(U)$, $k \in \Gamma(K)$,
 - (b) $[\Gamma(U), \Gamma(U)]_\Delta \subseteq \Gamma(U)$,
 - (c) $R_\Delta(U \otimes U \otimes (E \oplus T^*M)) \subseteq K$.

Proof. This is an immediate corollary of the results in the preceding subsection, using the fact that $R_\Delta((TM \oplus E^*) \otimes (TM \oplus E^*) \otimes (0 \oplus T^*M)) = 0$. (To see this, use Proposition 3.4 and the fact that the anchor is pr_{TM} .) \square

Corollary 4.23. *Let D be a sub-double vector bundle of $TE \oplus T^*E$ over $U \subseteq TM \oplus E^*$ and $K \subseteq E \oplus T^*M$ and choose a Dorfman connection Δ that is adapted to D . Then*

- (1) D is an isotropic subalgebroid of $TE \oplus T^*E \rightarrow E$ if and only if
 - (a) $U \subseteq K^\circ$,
 - (b) $\Delta_u k \in \Gamma(K)$ for all $u \in \Gamma(U)$, $k \in \Gamma(K)$,
 - (c) $(U, \text{pr}_{TM}|_U, [\cdot, \cdot]_\Delta|_{\Gamma(U) \times \Gamma(U)})$ is a skew-symmetric dull algebroid.
 - (d) the induced Dorfman connection

$$\tilde{\Delta} : \Gamma(U) \times \Gamma((E \oplus T^*M)/K) \rightarrow \Gamma((E \oplus T^*M)/K)$$

is flat.

- (2) D is a Dirac structure if and only if
 - (a) $U = K^\circ$,
 - (b) $\Delta_u k \in \Gamma(K)$ for all $u \in \Gamma(U)$, $k \in \Gamma(K)$,
 - (c) $(U, \text{pr}_{TM}|_U, [\cdot, \cdot]_\Delta|_{\Gamma(U) \times \Gamma(U)})$ is a Lie algebroid.

Note that in the second situation, the induced Dorfman connection $\bar{\Delta}$ is just the Lie derivative

$$\mathcal{L} = \bar{\Delta} : \Gamma(U) \times \Gamma(U^*) \rightarrow \Gamma(U^*),$$

which flatness is equivalent to the fact that the restriction of $\llbracket \cdot, \cdot \rrbracket_{\Delta}$ to $\Gamma(U)$ satisfies the Jacobi-identity.

Remark 4.24. (1) Note that $K = U^{\circ}$ and $\Delta_u k \in \Gamma(K)$ for all $u \in \Gamma(U)$, $k \in \Gamma(K)$ imply together that the dull bracket restricts to a bracket on $\Gamma(U)$.
 (2) Using the following Proposition 4.25 and Remark 4.7, it can be checked directly that if the conditions in (2) of Corollary 4.23 are satisfied for Δ , then they are also satisfied for any Δ' that is (U, K) -equivalent to Δ .
 (3) We will say that $(U, K, [\Delta])$ is a Dirac triple if the corresponding sub-double vector bundle $D_{(U, K, [\Delta])}$ is a Dirac structure on E . By the considerations above, we find that Dirac sub-double vector bundles of $TE \oplus T^*E \rightarrow E$ are in one-one correspondence with Dirac triples.

Proposition 4.25. *Let $E \rightarrow M$ be a vector bundle and choose a VB-triple $(U, K, [\Delta]_{U, K})$ such that $U = K^{\circ}$. Then for any two representatives $\Delta, \Delta' \in [\Delta]_{U, K}$, we have*

$$\llbracket u_1, u_2 \rrbracket_{\Delta} = \llbracket u_1, u_2 \rrbracket_{\Delta'}$$

for all $u_1, u_2 \in \Gamma(U)$.

Proof. Since $\text{pr}_{TM} \llbracket u_1, u_2 \rrbracket_{\Delta} = [\text{pr}_{TM} u_1, \text{pr}_{TM} u_2] = \text{pr}_{TM} \llbracket u_1, u_2 \rrbracket_{\Delta'}$, we need only to check that

$$\langle \llbracket u_1, u_2 \rrbracket_{\Delta}, (e, 0) \rangle = \langle \llbracket u_1, u_2 \rrbracket_{\Delta'}, (e, 0) \rangle$$

for all $e \in \Gamma(E)$. But this is immediate by the hypothesis, the duality of Δ and $\llbracket \cdot, \cdot \rrbracket_{\Delta}$ and the definition of (U, K) -equivalence. \square

We conclude this section with a study of our recurrent examples.

Example 4.26. In the situation of Example 4.16, choose two subbundles $F_M \subseteq TM$ and $C \subseteq E$. Set $U := F_M \oplus C^*$ and $K := C \oplus F_M^{\circ} = U^{\circ}$. The sub-double vector bundle $D_{U, K, \Delta}$ corresponding to U , K and the standard Dorfman connection associated to ∇ is then the direct sum of a subbundle $F_E \subseteq TM$, with $C_E \subseteq T^*E$. Since $U = K^{\circ}$, we get immediately that $C_E = F_E^{\circ}$ and $D_{(U, K, [\Delta])}$ is the trivial almost Dirac structure $F_E \oplus F_E^{\circ}$. An application of Corollary 4.23 to this situation yields that $F_E \oplus F_E^{\circ}$ is Dirac if and only if

- (1) F_M is involutive,
- (2) $\nabla_X c \in \Gamma(C)$ for all $X \in \Gamma(F_M)$ and $c \in \Gamma(C)$ and
- (3) the induced connection $\tilde{\nabla} : \Gamma(F_M) \times \Gamma(E/C) \rightarrow \Gamma(E/C)$ is flat.

Since $F_E \oplus F_E^{\circ}$ is Dirac if and only if $F_E \subseteq TE$ is involutive, we recover one of the results in [16].

Example 4.27. In the situation of Example 4.17, consider $U = \text{graph}(\rho : A \rightarrow TM)$ and $K = \text{graph}(-\rho^* : T^*M \rightarrow A^*) = U^{\circ}$. A straightforward computation shows that $\Delta_{(\rho(a), a)}(-\rho^*(\omega), \omega) = (-\rho^*(\nabla_a^{\text{bas}} \omega), \nabla_a^{\text{bas}} \omega) \in \Gamma(K)$ for all $a \in \Gamma(A)$ and $\omega \in \Omega^1(M)$. Furthermore, we have

$$\llbracket (\rho(a), a), (\rho(b), b) \rrbracket_{\Delta} = (\rho([a, b]), [a, b])$$

for all $a, b \in \Gamma(A)$, which shows that $(U, \text{pr}_{TM}, \llbracket \cdot, \cdot \rrbracket_{\Delta})$ is a Lie algebroid if and only if A is a Lie algebroid. We have:

$$\begin{aligned} \bar{\Delta}_{(\rho(a), a)} \overline{(\xi, 0)} &= \overline{\left(\langle \xi, \nabla^{\text{bas}} a \rangle + \nabla_{\rho(a)}^* \xi - \rho^* \langle \nabla \cdot a, \xi \rangle, \langle \nabla \cdot a, \xi \rangle \right)} \\ &= \overline{\left(\langle \xi, \nabla^{\text{bas}} a \rangle + \nabla_{\rho(a)}^* \xi, 0 \right)} = \overline{(\mathcal{L}_a \xi, 0)}. \end{aligned}$$

Finally, the right-hand side of (4.7) vanishes for $(\rho(a), a), (\rho(b), b), (\rho(c), c) \in \Gamma(U)$ and arbitrary (ξ, θ) if and only if A is a Lie algebroid.

Hence, we find that the sub-double vector bundle D of $TA^* \oplus T^*A^*$ associated to U, K and Δ is an almost Dirac structure on A^* and a Dirac structure if and only if A is a Lie algebroid. The vector bundle $D \rightarrow A^*$ is in fact the graph of the vector bundle morphism

$$\pi_A^\sharp : T^*A^* \rightarrow TA^*$$

associated to the linear almost Poisson structure defined on A^* by the skew-symmetric dull algebroid structure on A . Indeed, D is spanned by the sections k^\uparrow for $k \in \Gamma(K)$ and \tilde{u} for $u \in \Gamma(U)$, or, equivalently, by the sections

$$(-\rho^*\theta^\uparrow, q_{A^*}^*\theta)$$

for $\theta \in \Omega^1(M)$ and

$$(\widetilde{\rho(a)}, \tilde{a})$$

for $a \in \Gamma(A)$, where

$$\begin{aligned} \widetilde{\rho(a)}(\xi_m) &= T_m \xi(\rho(a)(m)) - \left. \frac{d}{dt} \right|_{t=0} \xi_m + t \cdot \mathcal{L}_a \xi(m) \\ \tilde{a}(\xi_m) &= \mathbf{d}_{\xi_m} l_a. \end{aligned}$$

But by Appendix A, these are exactly the sections $(\pi_A^\sharp(q_{A^*}^*\theta), q_{A^*}^*\theta)$ and $(\pi_A^\sharp(\mathbf{d}l_a), \mathbf{d}l_a)$.

Example 4.28. Consider, in the situation of Example 4.18, $U =: \text{graph}(-\sigma^* : TM \rightarrow E^*)$ and $K = \text{graph}(\sigma : E \rightarrow T^*M)$. Then $U = K^\circ$ by definition and since

$$\Delta_{(X, -\sigma^*X)}(e, \sigma(e)) = (\nabla_X e, \sigma(\nabla_X e))$$

by definition, we find that $\Delta_u k \in \Gamma(K)$ for all $u \in \Gamma(U)$ and $k \in \Gamma(K)$. Furthermore, we have $\llbracket (X, -\sigma^*X), (Y, -\sigma^*Y) \rrbracket_\Delta = ([X, Y], -\sigma^*[X, Y])$ for all $X, Y \in \mathfrak{X}(M)$ and U is a Lie algebroid (isomorphic to TM with the Lie bracket of vector fields). Alternatively, the Jacobiator in (4.8) is easily seen to vanish on this type of sections. This shows that the double vector subbundle $D \subseteq TE \oplus T^*E$ defined by U, K and Δ is a Dirac structure.

By the considerations in Appendix B, D is the graph of the vector bundle morphism $TE \rightarrow T^*E$ defined by the closed 2-form $\sigma^*\omega_{\text{can}}$.

Example 4.29. We now combine Examples 4.26 and 4.28 to recover an example in [18].

We consider the vector bundle $T^*M \rightarrow M$ endowed with a TM -connection ∇ and the Dorfman connection

$$\Delta : \Gamma(TM \oplus TM) \times \Gamma(T^*M \oplus T^*M) \rightarrow \Gamma(T^*M \oplus T^*M),$$

$$\Delta_{(X, Y)}(\theta, \omega) = (\nabla_X \theta, \mathcal{L}_X(\omega - \theta) + \langle \nabla^*(X + Y), \omega \rangle + \nabla_X \theta).$$

Consider a subbundle $F \subseteq TM$ and $U := \{(x, -x) \mid x \in F\} \subseteq TM \oplus TM$. The annihilator $K = U^\circ$ is then given by $K = \{(\alpha, \beta) \in T^* \oplus T^*M \mid \alpha - \beta \in F^\circ\}$.

Note that by Example 4.18, the dull bracket on $TM \oplus TM$ is skew-symmetric. In fact, it is easy to see that its restriction to U is just the Lie bracket of vector fields

$$\llbracket (X, -X), (Y, -Y) \rrbracket_\Delta = ([X, Y], -[X, Y])$$

for all $X, Y \in \Gamma(F)$. Hence, we know already that the sub-double vector bundle $D_{(U, K, [\Delta])}$ is an almost Dirac structure on T^*M . An easy computation using Appendix B yields that

$$D_{(U, K, [\Delta])}(\alpha) = \{(x_\alpha, \omega_{\text{can}}^\flat(x_\alpha) + \theta_\alpha) \mid x_\alpha \in \mathcal{F}(\alpha), \theta_\alpha \in \mathcal{F}^\circ(\alpha)\}$$

for all $\alpha \in T^*M$, where $\mathcal{F} = (Tc_M)^{-1}(F)$. Assume that M is the configuration space of a nonholonomic mechanical system and F the constraints distribution. If L is the Lagrangian

of the system, then the pullback of the Dirac structure that we find to the constraints submanifold $\mathbb{F}L(F) \subseteq T^*M$ is one of the frameworks proposed in [18] for the study of the nonholonomic system.

5. DORFMAN CONNECTIONS AND LIE ALGEBROIDS

We consider in this section a Lie algebroid $(A \rightarrow M, \rho, [\cdot, \cdot])$ and a Dorfman connection

$$\Delta : \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \rightarrow \Gamma(A \oplus T^*M)$$

with corresponding dull bracket $[\![\cdot, \cdot]\!]_\Delta$ and anchor pr_{TM} on $TM \oplus A^*$.

We will work here with the map

$$\Omega : \Gamma(TM \oplus A^*) \times \Gamma(A) \rightarrow \Gamma(A \oplus T^*M)$$

$$\Omega_{(X, \xi)} a = \Delta_{(X, \xi)}(a, 0) - (0, \mathbf{d}\langle \xi, a \rangle).$$

Ω has the following properties

- (1) $\Omega_{\varphi(X, \xi)} a = \varphi \Omega_{(X, \xi)} a$,
- (2) $\Omega_{(X, \xi)}(\varphi a) = \varphi \Omega_{(X, \xi)} a + X(\varphi)(a, 0) - \langle \xi, a \rangle(0, \mathbf{d}\varphi)$

for all $\varphi \in C^\infty(M)$, $a \in \Gamma(A)$ and $(X, \xi) \in \Gamma(TM \oplus A^*)$.

For each $a \in \Gamma(A)$, we have two derivations over $\rho(a)$:

$$\mathcal{L}_a : \Gamma(A \oplus T^*M) \rightarrow \Gamma(A \oplus T^*M),$$

$$\mathcal{L}_a(b, \theta) = ([a, b], \mathcal{L}_{\rho(a)}\theta)$$

and

$$\mathcal{L}_a : \Gamma(TM \oplus A^*) \rightarrow \Gamma(TM \oplus A^*)$$

$$\mathcal{L}_a(X, \xi) = ([\rho(a), X], \mathcal{L}_a \xi).$$

Note that

$$\mathcal{L}_{\varphi a}(b, \theta) = \varphi \mathcal{L}_a(b, \theta) + (-\rho(b)(\varphi)a, \langle \theta, \rho(a) \rangle \mathbf{d}\varphi).$$

Finally, note that there is a ‘‘Dorfman-like’’ bracket $[\cdot, \cdot]_D$ on sections of $A \oplus T^*M$:

$$[(a, \theta), (b, \omega)]_D = ([a, b], \mathcal{L}_{\rho(a)}\omega - \mathbf{i}_{\rho(b)}\mathbf{d}\theta)$$

for $(a, \theta), (b, \omega) \in \Gamma(A \oplus T^*M)$. We have

$$(5.9) \quad [\sigma_1, \sigma_2]_D + [\sigma_2, \sigma_1]_D = (0, \mathbf{d}\langle \sigma_1, (\rho, \rho^*)\sigma_2 \rangle)^3$$

and the Jacobi identity in Leibniz form

$$(5.10) \quad [\sigma_1, [\sigma_2, \sigma_3]] = [[\sigma_1, \sigma_2], \sigma_3] + [\sigma_2, [\sigma_1, \sigma_3]]$$

for all $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(A \oplus T^*M)$. Note that $(A \oplus T^*M, \rho \circ \text{pr}_A, [\cdot, \cdot]_D, \langle \cdot, \cdot \rangle_D)$ is not a Courant algebroid because $\langle \cdot, \cdot \rangle_D$ is in general degenerate.

³We will write $\langle \cdot, \cdot \rangle_D$ for this pairing of $A \oplus T^*M$ with itself, i.e. $\langle \sigma_1, \sigma_2 \rangle_D = \langle \sigma_1, (\rho, \rho^*)\sigma_2 \rangle$ for all $\sigma_1, \sigma_2 \in \Gamma(A \oplus T^*M)$.

5.1. The basic connections associated to Δ .

Proposition 5.1. *The two maps*

$$\begin{aligned}\nabla^{\text{bas}} : \Gamma(A) \times \Gamma(TM \oplus A^*) &\rightarrow \Gamma(TM \oplus A^*) \\ \nabla_a^{\text{bas}}(X, \xi) &= (\rho, \rho^*)(\Omega_{(X, \xi)}a) + \mathcal{L}_a(X, \xi)\end{aligned}$$

and

$$\begin{aligned}\nabla^{\text{bas}} : \Gamma(A) \times \Gamma(A \oplus T^*M) &\rightarrow \Gamma(A \oplus T^*M) \\ \nabla_a^{\text{bas}}(b, \theta) &= \Omega_{(\rho, \rho^*)(b, \theta)}a + \mathcal{L}_a(b, \theta)\end{aligned}$$

are connections in the usual sense.

Proof. Recall the notation $\mathbf{d}_{A^*}\varphi := \rho^*\mathbf{d}\varphi$ for all $\varphi \in C^\infty(M)$. We compute for $\varphi \in C^\infty(M)$, $a, b \in \Gamma(A)$, $\xi \in \Gamma(A^*)$, $\theta \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$:

$$\begin{aligned}\nabla_{\varphi a}^{\text{bas}}(X, \xi) &= (\rho, \rho^*)(\Omega_{(X, \xi)}(\varphi a)) + ([\varphi \rho(a), X], \mathcal{L}_{\varphi a}\xi) \\ &= \varphi \cdot \nabla_a^{\text{bas}}(X, \xi) + (\rho, \rho^*)(X(\varphi)a, -\langle \xi, a \rangle \mathbf{d}\varphi) + (-X(\varphi)\rho(a), \langle \xi, a \rangle \mathbf{d}_A^*\varphi) \\ &= \varphi \cdot \nabla_a^{\text{bas}}(X, \xi), \\ \nabla_{\varphi a}^{\text{bas}}(b, \theta) &= \Omega_{(\rho, \rho^*)(b, \theta)}(\varphi a) + \mathcal{L}_{\varphi a}(b, \theta) \\ &= \varphi \cdot \nabla_a^{\text{bas}}(b, \theta) + (\rho(b)(\varphi) \cdot a, 0) - \langle \theta, \rho(a) \rangle(0, \mathbf{d}\varphi) - (\rho(b)(\varphi)a, 0) + \langle \theta, \rho(a) \rangle(0, \mathbf{d}\varphi) \\ &= \varphi \cdot \nabla_a^{\text{bas}}(b, \theta), \\ \nabla_a^{\text{bas}}(\varphi(X, \xi)) &= (\rho, \rho^*)(\Omega_{\varphi(X, \xi)}a) + \mathcal{L}_a(\varphi(X, \xi)) = \varphi \nabla_a^{\text{bas}}(X, \xi) + \rho(a)(\varphi)(X, \xi) \\ \nabla_a^{\text{bas}}(\varphi(b, \theta)) &= \Omega_{(\rho, \rho^*)(\varphi(b, \theta))}a + \mathcal{L}_a(\varphi(b, \theta)) = \varphi \nabla_a^{\text{bas}}(b, \theta) + \rho(a)(\varphi)(b, \theta).\end{aligned}$$

□

The following proposition is easily checked, and shows that the connections are in general not dual to each other.

Proposition 5.2. *We have*

$$(5.11) \quad \langle \nabla_a^{\text{bas}}v, \sigma \rangle + \langle v, \nabla_a^{\text{bas}}\sigma \rangle = \rho(a)\langle v, \sigma \rangle - \langle \text{Skew}_\Delta(v, (\rho, \rho^*)\sigma), a \rangle$$

and

$$(5.12) \quad \nabla_a^{\text{bas}}(\rho, \rho^*)\sigma = (\rho, \rho^*)\nabla_a^{\text{bas}}\sigma$$

for all $a \in \Gamma(A)$, $v \in \Gamma(TM \oplus A^*)$ and $\sigma \in \Gamma(A \oplus T^*M)$.

Definition 5.3. *The connections in Proposition 5.1 will be called the **basic connections** associated to Δ . We will sometimes also write $\nabla_\sigma^{\text{bas}} := \nabla_{\text{pr}_A}^{\text{bas}}\sigma$ for a section $\sigma \in \Gamma(A \oplus T^*M)$.*

Proposition 5.4. *The map*

$$R_\Delta^{\text{bas}} : \Gamma(A) \times \Gamma(A) \times \Gamma(TM \oplus A^*) \rightarrow \Gamma(A \oplus T^*M)$$

given by

$$R_\Delta^{\text{bas}}(a, b)(X, \xi) = -\Omega_{(X, \xi)}[a, b] + \mathcal{L}_a(\Omega_{(X, \xi)}b) - \mathcal{L}_b(\Omega_{(X, \xi)}a) + \Omega_{\nabla_b^{\text{bas}}(X, \xi)}a - \Omega_{\nabla_a^{\text{bas}}(X, \xi)}b.$$

is tensorial, i.e. a section of

$$A^* \otimes A^* \otimes (A \oplus T^*M) \otimes (A \oplus T^*M).$$

Definition 5.5. *The tensor R_Δ^{bas} will be called the **basic curvature** associated to Δ .*

Proof of Proposition 5.4. We compute for $v = (X, \xi) \in \Gamma(TM \oplus A^*)$ and $a, b \in \Gamma(A)$:

$$\begin{aligned}
R_\Delta(\varphi a, b)v &= -\Omega_v(\varphi[a, b] - \rho(b)(\varphi)a) + \mathcal{L}_{\varphi a}\Omega_v b \\
&\quad - \mathcal{L}_b(\varphi\Omega_v a + X(\varphi)a - \langle \xi, a \rangle(0, \mathbf{d}\varphi)) + \Omega_{\nabla_b^{\text{bas}} v}(\varphi a) - \Omega_{\varphi \nabla_a^{\text{bas}} v} b \\
&= \varphi R_\Delta(a, b)v - X(\varphi)([a, b], 0) + \langle \xi, [a, b] \rangle(0, \mathbf{d}\varphi) + \rho(b)(\varphi)\Omega_v a \\
&\quad + X(\rho(b)(\varphi))(a, 0) - \langle \xi, a \rangle(0, \mathbf{d}(\rho(b)(\varphi))) \\
&\quad - ((\rho \circ \text{pr}_A)(\Omega_v b)(\varphi)a, -\langle \Omega_v b, (a, 0) \rangle \mathbf{d}\varphi) \\
&\quad - \rho(b)(\varphi)\Omega_v a - \rho(b)(X(\varphi))(a, 0) - X(\varphi)([b, a], 0) \\
&\quad + \rho(b)\langle \xi, a \rangle(0, \mathbf{d}\varphi) + \langle \xi, a \rangle(0, \mathcal{L}_{\rho(b)} \mathbf{d}\varphi) \\
&\quad + ((\rho \circ \text{pr}_A)(\Omega_v b) + [\rho(b), X])(\varphi)(a, 0) - \langle \Omega_v b, (a, 0) \rangle(0, \mathbf{d}\varphi) \\
&\quad - \langle \mathcal{L}_b \xi, a \rangle(0, \mathbf{d}\varphi) \\
&= \varphi R_\Delta(a, b)v, \\
R_\Delta(a, b)(\varphi v) &= \varphi R_\Delta(a, b)v - \rho(a)(\varphi)\Omega_v b + \rho(b)(\varphi)\Omega_v a - \rho(b)(\varphi)\Omega_v a + \rho(a)(\varphi)\Omega_v b \\
&= \varphi R_\Delta(a, b)v.
\end{aligned}$$

□

Proposition 5.6. *The basic curvature has the following properties:*

- (1) $R_{\nabla^{\text{bas}}} = R_\Delta^{\text{bas}} \circ (\rho, \rho^*)$,
- (2) $R_{\nabla^{\text{bas}}} = (\rho, \rho^*) \circ R_\Delta^{\text{bas}}$.

Proof. (1) For $\sigma \in \Gamma(A \oplus T^*M)$ and $a, b \in \Gamma(A)$, we have

$$\begin{aligned}
R_\Delta(a, b)((\rho, \rho^*)\sigma) &= -\Omega_{(\rho, \rho^*)\sigma}[a, b] + \mathcal{L}_a(\Omega_{(\rho, \rho^*)\sigma} b) - \mathcal{L}_b(\Omega_{(\rho, \rho^*)\sigma} a) \\
&\quad + \Omega_{\nabla_b^{\text{bas}}(\rho, \rho^*)\sigma} a - \Omega_{\nabla_a^{\text{bas}}(\rho, \rho^*)\sigma} b \\
&= -\Omega_{(\rho, \rho^*)\sigma}[a, b] - \mathcal{L}_{[a, b]}\sigma + \mathcal{L}_a(\Omega_{(\rho, \rho^*)\sigma} b + \mathcal{L}_b\sigma) - \mathcal{L}_b(\Omega_{(\rho, \rho^*)\sigma} a + \mathcal{L}_a\sigma) \\
&\quad + \Omega_{\nabla_b^{\text{bas}}(\rho, \rho^*)\sigma} a - \Omega_{\nabla_a^{\text{bas}}(\rho, \rho^*)\sigma} b \\
&= -\nabla_{[a, b]}^{\text{bas}}\sigma + \nabla_a^{\text{bas}}\nabla_b^{\text{bas}}\sigma - \nabla_b^{\text{bas}}\nabla_a^{\text{bas}}\sigma = R_{\nabla^{\text{bas}}}(a, b)\sigma.
\end{aligned}$$

- (2) The second equality is shown in the same manner.

□

5.2. The Lie algebroid structure on $TA \oplus T^*A \rightarrow TM \oplus A^*$. Consider a Lie algebroid A and a Dorfman connection

$$\Delta : \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \rightarrow \Gamma(A \oplus T^*M).$$

Then, for any section $a \in \Gamma(A)$, we define

$$\Sigma_a \in \Gamma_{TM \oplus A^*}(TA \oplus T^*A)$$

by

$$\Sigma_a(v_m, \xi_m) = (T_m a v_m, \mathbf{d}_{a_m} \ell_\xi) - \Delta_{(X, \xi)}(a, 0)^\dagger(a_m)$$

for any choice of section $(X, \xi) \in \Gamma(TM \oplus A^*)$ such that $(X, \xi)(m) = (v_m, \xi_m)$. That is, we have

$$\Sigma_a = (Ta, R(\mathbf{d}a)) - \Omega.a^\dagger = \tilde{a} - \Omega.a^\dagger$$

for all $a \in \Gamma(A)$ (see the description of the Lie algebroid structure on $T^*A \rightarrow A^*$ in Appendix D).

Theorem 5.7. *The Lie algebroid structure on $TA \oplus T^*A \rightarrow TM \oplus A^*$ can be characterized as follows, where we write $\Theta : TA \oplus T^*A \rightarrow T(TM \oplus A^*)$ for its anchor:*

- (1) $[\Sigma_a, \Sigma_b] = \Sigma_{[a, b]} - R_\Delta^{\text{bas}}(a, b)^\dagger$,

- (2) $[\Sigma_a, \sigma^\dagger] = (\nabla_a^{\text{bas}} \sigma)^\dagger,$
- (3) $[\sigma_1^\dagger, \sigma_2^\dagger] = 0,$
- (4) $\Theta(\Sigma_a) = \widetilde{\nabla_a^{\text{bas}}} \in \mathfrak{X}(TM \oplus A^*),$
- (5) $\Theta(\sigma^\dagger) = ((\rho, \rho^*)\sigma)^\dagger \in \mathfrak{X}(TM \oplus A^*).$

Remark 5.8. In other words, $(\rho, \rho^*) : A \oplus T^*M \rightarrow TM \oplus A^*$, the basic connections ∇^{bas} and the basic curvature R_Δ^{bas} define the representation up to homotopy describing the VB-Lie algebroid structure on $TA \oplus T^*A \rightarrow TM \oplus A^*$ in terms of the splitting given by Δ (see [10]).

Proof of Theorem 5.7. The proof of this theorem is just checking of the formulas, using the description of the Lie algebroid structure on $TA \oplus T^*A \rightarrow TM \oplus A^*$ that can be found in Appendix D.

We start with the Lie algebroid brackets. Choose $a, b \in \Gamma(A)$ and $\sigma \in \Gamma(A \oplus T^*M)$. We have, using Proposition D.1:

(1)

$$\begin{aligned} [\Sigma_a, \Sigma_b] &= [\tilde{a} - \Omega.a^\dagger, \tilde{b} - \Omega.b^\dagger] \\ &= \widetilde{[a, b]} - (\mathcal{L}_a \Omega.b)^\dagger + (\mathcal{L}_b \Omega.a)^\dagger + (\Omega.b \circ (\rho, \rho^*) \circ \Omega.a - \Omega.a \circ (\rho, \rho^*) \circ \Omega.b)^\dagger \\ &= \Sigma_{[a, b]} - R_\Delta(a, b)^\dagger, \end{aligned}$$

since we have, for all $v \in \Gamma(TM \oplus A^*)$:

$$\begin{aligned} & -(\mathcal{L}_a \Omega.b)(v) + (\mathcal{L}_b \Omega.a)(v) + (\Omega.b \circ (\rho, \rho^*) \circ \Omega.a)(v) - (\Omega.a \circ (\rho, \rho^*) \circ \Omega.b)(v) \\ &= -\mathcal{L}_a \Omega_v b + \Omega_{\mathcal{L}_a v} b + \mathcal{L}_b \Omega_v a - \Omega_{\mathcal{L}_b v} a + \Omega_{(\rho, \rho^*) \Omega_v a} b - \Omega_{(\rho, \rho^*) \Omega_v b} a \\ &= -\mathcal{L}_a \Omega_v b + \mathcal{L}_b \Omega_v a + \Omega_{\nabla_a^{\text{bas}} v} b - \Omega_{\nabla_b^{\text{bas}} v} a = -R_\Delta(a, b)v - \Omega_v[a, b]. \end{aligned}$$

$$(2) \quad [\Sigma_a, \sigma^\dagger] = (\mathcal{L}_a \sigma)^\dagger + \Omega_{(\rho, \rho^*) \sigma} a^\dagger = (\nabla_a^{\text{bas}} \sigma)^\dagger.$$

For the anchor map, we compute:

$$(4) \quad \Theta(\Sigma_a)(\ell_\sigma) = \ell_{\mathcal{L}_a \sigma - (\Omega.a)^*((\rho, \rho^*)\sigma)}, \text{ which yields the desired equality since}$$

$$\begin{aligned} \langle (\Omega.a)^*((\rho, \rho^*)\sigma), v \rangle &= \langle \Omega_v a, (\rho, \rho^*)\sigma \rangle = \langle (\rho, \rho^*)\Omega_v a, \sigma \rangle = \langle \nabla_a^{\text{bas}} v - \mathcal{L}_a v, \sigma \rangle \\ &= \rho(a) \langle v, \sigma \rangle - \langle \text{Skew}_\Delta(v, (\rho, \rho^*)\sigma), a \rangle - \langle v, \nabla_a^{\text{bas}} \sigma \rangle - \langle \mathcal{L}_a v, \sigma \rangle \\ &= -\langle \text{Skew}_\Delta(v, (\rho, \rho^*)\sigma), a \rangle - \langle v, \nabla_a^{\text{bas}} \sigma \rangle + \langle v, \mathcal{L}_a \sigma \rangle \end{aligned}$$

and consequently

$$\langle v, \mathcal{L}_a \sigma - (\Omega.a)^*((\rho, \rho^*)\sigma) \rangle = \langle v, \nabla_a^{\text{bas}*} \sigma \rangle$$

by (1) of Proposition 5.2.

The remaining equalities follow from Proposition D.1. □

Theorem 5.9. Consider a Lie algebroid A and a Dorfman connection

$$\Delta : \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \rightarrow \Gamma(A \oplus T^*M).$$

Let $U \subseteq TM \oplus A^*$ and $K \subseteq A \oplus T^*M$ be subbundles. Then the sub-double vector bundle $D_{(U, K, [\Delta])}$ is a subalgebroid of $TA \oplus T^*A \rightarrow TM \oplus A^*$ over U if and only if:

- (1) $(\rho, \rho^*)(K) \subseteq U,$
- (2) $\nabla_a^{\text{bas}} k \in \Gamma(K)$ for all $a \in \Gamma(A)$ and $k \in \Gamma(K),$
- (3) $\nabla_a^{\text{bas}} u \in \Gamma(U)$ for all $a \in \Gamma(A)$ and $u \in \Gamma(U),$
- (4) $R_\Delta^{\text{bas}}(a, b)u \in \Gamma(K)$ for all $u \in \Gamma(U), a, b \in \Gamma(A).$

Proof. Assume that $D_{(U,K,[\Delta])} \rightarrow U$ is a subalgebroid of $TA \oplus T^*A \rightarrow TM \oplus A^*$. Then we have $((\rho, \rho^*)k)^\dagger|_U = \Theta(k^\dagger|_U) \in \mathfrak{X}(U)$ and $\widehat{\nabla}_a^{\text{bas}}|_U = \Theta(\tilde{a}|_U) \in \mathfrak{X}(U)$ for all $a \in \Gamma(A)$ and $k \in \Gamma(K)$. This is the case if and only if $((\rho, \rho^*)k)^\dagger(\ell_l)|_U = 0$ and $\widehat{\nabla}_a^{\text{bas}}(\ell_l)|_U = 0$ for all $l \in \Gamma(U^\circ)$. Since $((\rho, \rho^*)k)^\dagger(\ell_l) = \pi^*\langle(\rho, \rho^*)k, l\rangle$ and $\widehat{\nabla}_a^{\text{bas}}(\ell_l) = \ell_{\nabla_a^{\text{bas}}l}$, we find that $(\rho, \rho^*)k$ must be a section of U and $\nabla_a^{\text{bas}}l \in \Gamma(U^\circ)$ for all $l \in \Gamma(U^\circ)$. But the latter is equivalent to $\nabla_a^{\text{bas}}u \in \Gamma(U)$ for all $u \in \Gamma(U)$. We have in the same manner $(\nabla_a^{\text{bas}}k)^\dagger|_U = [\tilde{a}, k^\dagger]|_U \in \Gamma(D_{(U,K,[\Delta])})$ and $([\tilde{a}, b] - R_\Delta^{\text{bas}}(a, b)^\dagger)|_U = [\tilde{a}, \tilde{b}]|_U \in \Gamma(D_{(U,K,[\Delta])})$ for all $a, b \in \Gamma(A)$ and $k \in \Gamma(K)$. But this is only the case if $\nabla_a^{\text{bas}}k \in \Gamma(K)$ and, since $[\tilde{a}, b]|_U \in \Gamma(D_{(U,K,[\Delta])})$, $R_\Delta^{\text{bas}}(a, b)^\dagger|_U \in \Gamma(D_{(U,K,[\Delta])})$. The latter holds only if $R_\Delta^{\text{bas}}(a, b)u \in \Gamma(K)$ for all $u \in \Gamma(U)$.

The converse implication is shown in the same manner. \square

5.3. \mathcal{LA} -Dirac structures in $TA \oplus T^*A$. In this subsection and the next, we will study in more details the triples $(U, K, [\Delta]_{U,K})$ associated to Dirac structures on A that are at the same time Lie subalgebroids of $TA \oplus T^*A \rightarrow TM \oplus A^*$. We call such a Dirac structure an \mathcal{LA} -Dirac structure on A .

Theorem 5.10. *Consider a Lie algebroid A and a Dorfman connection*

$$\Delta : \Gamma(TM \oplus A^*) \times \Gamma(A \oplus T^*M) \rightarrow \Gamma(A \oplus T^*M).$$

Let $U \subseteq TM \oplus A^$ and $K \subseteq A \oplus T^*M$ be subbundles. Then $D_{(U,K,[\Delta])}$ is a Dirac structure in $TA \oplus T^*A \rightarrow A$ and a subalgebroid of $TA \oplus T^*A \rightarrow TM \oplus A^*$ over U if and only if $(U, K, [\Delta])$ is a \mathcal{LA} -Dirac triple, i.e. if and only if:*

- (1) $K = U^\circ$
- (2) $(\rho, \rho^*)(K) \subseteq U$,
- (3) $(U, \text{pr}_{TM}, [\cdot, \cdot]_\Delta)$ is a Lie algebroid,
- (4) $\nabla_a^{\text{bas}}k \in \Gamma(K)$ for all $a \in \Gamma(A)$ and $k \in \Gamma(K)$,
- (5) $R_\Delta^{\text{bas}}(a, b)u \in \Gamma(K)$ for all $u \in \Gamma(U)$, $a, b \in \Gamma(A)$.

Proof. This theorem follows from (2) in Corollary 4.23 and Theorem 5.9. Note that if $U = K^\circ$, $(\rho, \rho^*)K \subseteq U$ and $(U, \text{pr}_{TM}, [\cdot, \cdot]_\Delta)$ is a Lie algebroid, then ∇_a^{bas} preserves $\Gamma(U)$ if and only if ∇_a^{bas} preserves $\Gamma(K)$. So (2) and (3) in Theorem 5.9 become the same condition. \square

Remark 5.11. A triple $(U, K, [\Delta]_{U,K})$ satisfying (1)–(5) in Theorem 5.9 will be called an \mathcal{LA} -Dirac triple on A . We have found a one-one correspondence between \mathcal{LA} -Dirac triples on A and \mathcal{LA} -Dirac structures on the Lie algebroid A .

Example 5.12. Consider again Examples 4.17 and 4.27. Assume that A^* has itself also a Lie algebroid structure with anchor ρ_* and bracket $[\cdot, \cdot]_*$. For simplicity, we switch the roles of A and A^* in Examples 4.17 and 4.27. Here, the second condition is equivalent to

$$(5.13) \quad \rho_* \circ \rho^* = -\rho \circ \rho^*$$

We assume in the following that this condition is satisfied.

We have also:

$$\Omega_{(\rho_*(\xi), \xi)}a = (\mathcal{L}_\xi a - \rho_*^*\langle \nabla^*\xi, a \rangle, \langle \nabla^*\xi, a \rangle) - (0, \mathbf{d}\langle \xi, a \rangle) = (\mathbf{i}_\xi \mathbf{d}_A a + \rho_*^*\langle \xi, \nabla \cdot a \rangle, -\langle \xi, \nabla \cdot a \rangle),$$

for all $\xi \in \Gamma(A^*)$ and $a \in \Gamma(A)$ and so

$$\Omega_{(\rho_*(\rho^*\theta), \rho^*\theta)}a = \Omega_{(\rho_*(\rho^*\theta), \rho^*\theta)}a = (\mathbf{i}_{\rho^*\theta} \mathbf{d}_A a + \rho_*^*\langle \rho^*\theta, \nabla \cdot a \rangle, -\langle \rho^*\theta, \nabla \cdot a \rangle).$$

for all $\theta \in \Omega^1(M)$. In particular, if $\theta = \mathbf{d}\varphi$ for some $\varphi \in C^\infty(M)$, we get:

$$\begin{aligned} \nabla_a^{\text{bas}}(-\rho_*^*\mathbf{d}\varphi, \mathbf{d}\varphi) &= \Omega_{(\rho_*(\rho^*\mathbf{d}\varphi), \rho^*\mathbf{d}\varphi)}a + \mathcal{L}_a(-\rho_*^*\mathbf{d}\varphi, \mathbf{d}\varphi) \\ &= (\mathbf{i}_{\rho^*\mathbf{d}\varphi} \mathbf{d}_A a + \rho_*^*\langle \rho^*\mathbf{d}\varphi, \nabla \cdot a \rangle - [a, \rho_*^*\mathbf{d}\varphi], -\langle \rho^*\mathbf{d}\varphi, \nabla \cdot a \rangle + \mathcal{L}_{\rho(a)}(\mathbf{d}\varphi)) \\ &= (\mathbf{i}_{\mathbf{d}_{A^*}\varphi} \mathbf{d}_A a + \rho_*^*\langle \mathbf{d}_{A^*}\varphi, \nabla \cdot a \rangle - [a, \mathbf{d}_A \varphi], -\langle \mathbf{d}_{A^*}\varphi, \nabla \cdot a \rangle + \mathbf{d}(\rho(a)(\varphi))). \end{aligned}$$

Thus, using the first condition, $\nabla_a^{\text{bas}}(-\rho_*^* \mathbf{d}\varphi, \mathbf{d}\varphi) \in \Gamma(K)$ if and only if

$$\langle (\mathbf{i}_{\mathbf{d}_{A^*}\varphi} \mathbf{d}_A a + \rho_*^* \langle \mathbf{d}_{A^*}\varphi, \nabla \cdot a \rangle - [a, \mathbf{d}_A \varphi], -\langle \mathbf{d}_{A^*}\varphi, \nabla \cdot a \rangle + \mathbf{d}(\rho(a)(\varphi))) , (\rho_* \xi, \xi) \rangle = 0$$

for all $\xi \in \Gamma(A^*)$. But this pairing equals

$$\begin{aligned} & \langle (\mathbf{i}_{\mathbf{d}_{A^*}\varphi} \mathbf{d}_A a + \rho_*^* \langle \mathbf{d}_{A^*}\varphi, \nabla \cdot a \rangle - [a, \mathbf{d}_A \varphi], -\langle \mathbf{d}_{A^*}\varphi, \nabla \cdot a \rangle + \mathbf{d}(\rho(a)(\varphi))) , (\rho_* \xi, \xi) \rangle \\ &= -\langle \mathbf{d}_{A^*}\varphi, \nabla_{\rho_*(\xi)} a \rangle + \rho_*(\xi) \rho(a)(\varphi) + (\mathbf{d}_A a)(\mathbf{d}_{A^*}\varphi, \xi) + \langle \mathbf{d}_{A^*}\varphi, \nabla_{\rho_*(\xi)} a \rangle - \langle \xi, [a, \mathbf{d}_A \varphi] \rangle \\ &= \rho_*(\xi) \rho(a)(\varphi) + \rho_*(\mathbf{d}_{A^*}\varphi) \langle \xi, a \rangle - \rho_*(\xi) \langle \mathbf{d}_{A^*}\varphi, a \rangle - \langle a, [\mathbf{d}_{A^*}\varphi, \xi]_* \rangle - \langle \xi, [a, \mathbf{d}_A \varphi] \rangle \\ &= \langle (\rho_* \circ \rho^*)(\mathbf{d}\varphi), \mathbf{d}(\langle \xi, a \rangle) \rangle + \rho_*(\xi) \langle a, \mathbf{d}_{A^*}\varphi \rangle - \langle \mathcal{L}_\xi a, \mathbf{d}_{A^*}\varphi \rangle - \rho(a) \langle \xi, \mathbf{d}_A \varphi \rangle + \langle \mathcal{L}_a \xi, \mathbf{d}_A \varphi \rangle \\ &= \langle \mathbf{d}\varphi, (\rho \circ \rho_*^*) \mathbf{d}(\langle \xi, a \rangle) \rangle + \rho_*(\xi) \rho(a)(\varphi) - \rho(\mathcal{L}_\xi a)(\varphi) - \rho(a) \rho_*(\xi)(\varphi) + \rho_*(\mathcal{L}_a \xi)(\varphi) \\ &= ([\rho_*(\xi), \rho(a)] + \rho_*(\mathcal{L}_a \xi) - \rho(\mathcal{L}_\xi a) + \rho(\mathbf{d}_A \langle \xi, a \rangle))(\varphi). \end{aligned}$$

Since φ was arbitrary, we have shown that the fourth condition is satisfied if and only if

$$(5.14) \quad [\rho(a), \rho_*(\xi)] - \rho_*(\mathcal{L}_a \xi) + \rho(\mathcal{L}_\xi a) = \rho(\mathbf{d}_A \langle \xi, a \rangle)$$

for all $a \in \Gamma(A)$ and $\xi \in \Gamma(A^*)$. Thus, we have found until here (5.13) and (5.14), which are shown in [22, Theorem 12.1.9] to imply the fact that (A, A^*) is a Lie bialgebroid.

We conclude by showing that the last condition, on the basic curvature, follows as well from (5.13) and (5.14). Since $\Omega_{(\rho_*(\xi), \xi)} a = (\mathbf{i}_\xi \mathbf{d}_A a, 0) - (-\rho_*^* \langle \xi, \nabla \cdot a \rangle, \langle \xi, \nabla \cdot a \rangle)$ and $K = U^\circ$, we find

$$\langle \Omega_{(\rho_*(\xi), \xi)} a, (\rho_* \eta, \eta) \rangle = (\mathbf{d}_A a)(\xi, \eta)$$

for all $a \in \Gamma(A)$, $\xi, \eta \in \Gamma(A^*)$. The fourth condition together with the first identity in Proposition 5.2 and the first and third conditions imply that $\nabla_a^{\text{bas}} u \in \Gamma(A)$ for all $u \in \Gamma(U)$. Hence,

$$\begin{aligned} \nabla_a^{\text{bas}}(\rho_*(\xi), \xi) &= (\rho, \rho^*) (\mathbf{i}_\xi \mathbf{d}_A a + \rho_*^* \langle \xi, \nabla \cdot a \rangle, -\langle \xi, \nabla \cdot a \rangle) + \mathcal{L}_a(\rho_*(\xi), \xi) \\ &= (\rho_*(-\rho^* \langle \xi, \nabla \cdot a \rangle + \mathcal{L}_a \xi), -\rho^* \langle \xi, \nabla \cdot a \rangle + \mathcal{L}_a \xi) \end{aligned}$$

and

$$\langle \Omega_{\nabla_a^{\text{bas}}(\rho_*(\xi), \xi)} b, (\rho_* \eta, \eta) \rangle = (\mathbf{d}_A b)(-\rho^* \langle \xi, \nabla \cdot a \rangle + \mathcal{L}_a \xi, \eta)$$

for all $a, b \in \Gamma(A)$, $\xi, \eta \in \Gamma(A^*)$.

We get hence that

$$\begin{aligned}
\langle R_{\Delta}^{\text{bas}}(a, b)(\rho_*\xi, \xi), (\rho_*\eta, \eta) \rangle &= (\mathbf{d}_A[a, b])(\xi, \eta) - \langle \mathcal{L}_a(\mathbf{i}_{\xi}\mathbf{d}_Ab + \rho_*^*\langle \xi, \nabla.b \rangle, -\langle \xi, \nabla.b \rangle), (\rho_*\eta, \eta) \rangle \\
&\quad + \langle \mathcal{L}_b(\mathbf{i}_{\xi}\mathbf{d}_Aa + \rho_*^*\langle \xi, \nabla.a \rangle, -\langle \xi, \nabla.a \rangle), (\rho_*\eta, \eta) \rangle \\
&\quad - (\mathbf{d}_Aa)(-\rho_*^*\langle \xi, \nabla.b \rangle + \mathcal{L}_b\xi, \eta) + (\mathbf{d}_Ab)(-\rho_*^*\langle \xi, \nabla.a \rangle + \mathcal{L}_a\xi, \eta) \\
&= (\mathbf{d}_A[a, b])(\xi, \eta) - \rho(a)(\mathbf{d}_Ab(\xi, \eta)) + \mathbf{d}_Ab(\xi, \mathcal{L}_a\eta) \\
&\quad + \langle (\rho_*^*\langle \xi, \nabla.b \rangle, -\langle \xi, \nabla.b \rangle), \mathcal{L}_a(\rho_*\eta, \eta) \rangle \\
&\quad + \rho(b)(\mathbf{d}_Aa(\xi, \eta)) - \mathbf{d}_Aa(\xi, \mathcal{L}_b\eta) - \langle (\rho_*^*\langle \xi, \nabla.a \rangle, -\langle \xi, \nabla.a \rangle), \mathcal{L}_b(\rho_*\eta, \eta) \rangle \\
&\quad - (\mathbf{d}_Aa)(-\rho_*^*\langle \xi, \nabla.b \rangle + \mathcal{L}_b\xi, \eta) + (\mathbf{d}_Ab)(-\rho_*^*\langle \xi, \nabla.a \rangle + \mathcal{L}_a\xi, \eta) \\
&= (\mathbf{d}_A[a, b] - [a, \mathbf{d}_Ab] + [b, \mathbf{d}_Aa])(\xi, \eta) \\
&\quad + \langle \xi, \nabla_{\rho_*(\mathcal{L}_a\eta)}b \rangle - \langle \xi, \nabla_{[\rho(a), \rho_*\eta]}b \rangle \\
&\quad - \langle \xi, \nabla_{\rho_*(\mathcal{L}_b\eta)}a \rangle + \langle \xi, \nabla_{[\rho(b), \rho_*\eta]}a \rangle \\
&\quad + \rho_*\rho^*\langle \xi, \nabla.b \rangle(\langle a, \eta \rangle) - \rho_*\eta\langle \xi, \nabla_{\rho(a)}b \rangle + \langle a, [\eta, \rho^*\langle \xi, \nabla.b \rangle]_* \rangle \\
&\quad - \rho_*\rho^*\langle \xi, \nabla.a \rangle(\langle b, \eta \rangle) + \rho_*\eta\langle \xi, \nabla_{\rho(b)}a \rangle - \langle b, [\eta, \rho^*\langle \xi, \nabla.a \rangle]_* \rangle \\
&= (\mathbf{d}_A[a, b] - [a, \mathbf{d}_Ab] + [b, \mathbf{d}_Aa])(\xi, \eta) \\
&\quad + \langle \xi, \nabla_{\rho_*(\mathcal{L}_a\eta)}b \rangle - \langle \xi, \nabla_{[\rho(a), \rho_*\eta]}b \rangle \\
&\quad - \langle \xi, \nabla_{\rho_*(\mathcal{L}_b\eta)}a \rangle + \langle \xi, \nabla_{[\rho(b), \rho_*\eta]}a \rangle \\
&\quad + \langle \xi, \nabla_{\rho(\mathbf{d}_A\langle a, \eta \rangle)}b \rangle - \langle \xi, \nabla_{\rho(\mathcal{L}_\eta a)}b \rangle \\
&\quad - \langle \xi, \nabla_{\rho(\mathbf{d}_A\langle b, \eta \rangle)}a \rangle + \langle \xi, \nabla_{\rho(\mathcal{L}_\eta b)}a \rangle.
\end{aligned}$$

By (5.14), the second and the fourth lines, and the third and the fifth lines cancel each other. We find hence that the last condition is satisfied if and only if (A, A^*) is a Lie bialgebroid. Hence, $(U, K, [\Delta])$ is an \mathcal{LA} -Dirac triple if and only if (A, A^*) is a Lie bialgebroid, and so the graph of π_A is mroptic and Dirac if and only if (A, A^*) is a Lie bialgebroid. This recovers a result in [25].

Example 5.13. In the situation of Examples 4.18 and 4.28, assume furthermore that $E =: A$ is a Lie algebroid. To avoid confusions, we write ∇^A for the A -basic connections induced on A and TM by the Lie algebroid structure on A and the connection ∇ , and R_{∇}^A for the basic curvature associated to it.

The second condition of the last theorem reads here

$$(\rho, \rho^*)(a, \sigma(a)) = (\rho(a), -\sigma^*\rho(a))$$

for all $a \in \Gamma(A)$, i.e.

$$\rho^* \circ \sigma = -\sigma^* \circ \rho.$$

This is exactly the first axiom defining an IM-2-form $\sigma : A \rightarrow T^*M$ [6, 5]:

$$\langle \sigma(a), \rho(b) \rangle = -\langle \rho(a), \sigma(b) \rangle$$

for all $a, b \in \Gamma(A)$. We next compute $\nabla_a^{\text{bas}}(b, \sigma(b))$. We have

$$\Omega_{(X, -\sigma^*X)}a = (\nabla_Xa, -\mathcal{L}_X\sigma(a) + \sigma(\nabla_Xa)) + (0, \mathbf{d}\langle \sigma(a), X \rangle) = (\nabla_Xa, -\mathbf{i}_X\mathbf{d}\sigma(a) + \sigma(\nabla_Xa))$$

and as a consequence

$$\nabla_a^{\text{bas}}(b, \sigma(b)) = \Omega_{(\rho, \rho^*)(b, \sigma(b))}a + \mathcal{L}_a(b, \sigma(b)) = (\nabla_{\rho(b)}a + [a, b], \mathcal{L}_{\rho(a)}\sigma(b) - \mathbf{i}_{\rho(b)}\mathbf{d}\sigma(a) + \sigma(\nabla_{\rho(b)}a)).$$

We find hence that $\nabla_a^{\text{bas}}(b, \sigma(b)) \in \Gamma(K)$ if and only if $([a, b], \mathcal{L}_{\rho(a)}\sigma(b) - \mathbf{i}_{\rho(b)}\mathbf{d}\sigma(a)) \in \Gamma(K)$, i.e. if and only if

$$\sigma([a, b]) = \mathcal{L}_{\rho(a)}\sigma(b) - \mathbf{i}_{\rho(b)}\mathbf{d}\sigma(a).$$

Since this is the second axiom in the definition of an IM-2-form, we recover the fact that the graph of $(\sigma^*\omega_{\text{can}})^b : TA \rightarrow T^*A$ is a subalgebroid of $TA \oplus T^*A \rightarrow TM \oplus A^*$ over

$U = \text{graph}(-\sigma^*)$ only if $\sigma : A \rightarrow T^*M$ is an IM-2-form. To show the equivalence, we show that the last condition in the last theorem follows here again from the four other. We have, for $a, b \in \Gamma(A)$ and $X, Y \in \mathfrak{X}(M)$:

$$\Omega_{(X, -\sigma^*X)}a = -(0, \mathbf{i}_X \mathbf{d}\sigma(a)) + (\nabla_X a, \sigma(\nabla_X a)),$$

hence

$$\begin{aligned} \mathcal{L}_b \Omega_{(X, -\sigma^*X)}a &= -(0, \mathcal{L}_{\rho(b)} \mathbf{i}_X \mathbf{d}\sigma(a)) + ([b, \nabla_X a], \mathcal{L}_{\rho(b)} \sigma(\nabla_X a)) \\ &= -(0, \mathbf{i}_{[\rho(b), X]} \mathbf{d}\sigma(a) + \mathbf{i}_X \mathcal{L}_{\rho(b)} \mathbf{d}\sigma(a)) + ([b, \nabla_X a], \sigma([b, \nabla_X a]) + \mathbf{i}_{\rho(\nabla_X a)} \mathbf{d}\sigma(b)) \end{aligned}$$

and

$$\nabla_a^{\text{bas}}(X, -\sigma^*X) = -(\rho, \rho^*)(0, \mathbf{i}_X \mathbf{d}\sigma(a)) + (\rho, \rho^*)(\nabla_X a, \sigma(\nabla_X a)) + \mathcal{L}_a(X, -\sigma^*X)$$

which equals $(\nabla_a^A X, -\sigma^*(\nabla_a^A X))$ since $\nabla_a^{\text{bas}} u \in \Gamma(U)$ for all $u \in \Gamma(U)$.

We get hence

$$\begin{aligned} R_{\Delta}^{\text{bas}}(a, b)(X, -\sigma^*X) &= (R_{\nabla}^A(a, b)(X), \sigma(R_{\nabla}^A(a, b)(X))) \\ &\quad + (0, -\mathbf{i}_X \mathbf{d}\sigma([a, b]) + \mathbf{i}_{[\rho(a), X]} \mathbf{d}\sigma(b) + \mathbf{i}_X \mathcal{L}_{\rho(a)} \mathbf{d}\sigma(b) - \mathbf{i}_{\rho(\nabla_X b)} \mathbf{d}\sigma(a) \\ &\quad - \mathbf{i}_{[\rho(b), X]} \mathbf{d}\sigma(a) - \mathbf{i}_X \mathcal{L}_{\rho(b)} \mathbf{d}\sigma(a) + \mathbf{i}_{\rho(\nabla_X a)} \mathbf{d}\sigma(b) + \mathbf{i}_{\nabla_b^A X} \mathbf{d}\sigma(a) - \mathbf{i}_{\nabla_a^A X} \mathbf{d}\sigma(b)) \\ &= (R_{\nabla}^A(a, b)(X), \sigma(R_{\nabla}^A(a, b)(X))) - (0, \mathbf{i}_X \mathbf{d}(\sigma([a, b]) - \mathcal{L}_{\rho(a)} \sigma(b) + \mathbf{i}_{\rho(b)} \mathbf{d}\sigma(a))) \\ &= (R_{\nabla}^A(a, b)(X), \sigma(R_{\nabla}^A(a, b)(X))) \in \Gamma(K). \end{aligned}$$

We continue with the study of \mathcal{LA} -Dirac triples. We first observe that if (U, K, Δ) is a \mathcal{LA} -Dirac triple, then K inherits a Lie algebroid structure.

Theorem 5.14. *Consider an \mathcal{LA} -Dirac triple $(U, K, [\Delta]_{U, K})$. Then*

$$(K, \rho_K := \rho \circ \text{pr}_A, [\cdot, \cdot]_D|_{\Gamma(K) \times \Gamma(K)})$$

is a Lie algebroid and the map $(\rho, \rho^) : K \rightarrow U$ is a Lie algebroid morphism.*

We need the following two lemmas, which will also be useful later.

Lemma 5.15. *The equality*

$$(5.15) \quad \nabla_{\sigma_1}^{\text{bas}} \sigma_2 = -[\sigma_2, \sigma_1]_D + \Delta_{(\rho, \rho^*)(\sigma_2)} \sigma_1$$

*holds for all $\sigma_1, \sigma_2 \in \Gamma(A \oplus T^*M)$.*

Proof. Write $\sigma_1 = (a_1, \theta_2)$ and $\sigma_2 = (a_2, \theta_2) \in \Gamma(A \oplus T^*M)$. Then:

$$\begin{aligned} \nabla_{\sigma_1}^{\text{bas}} \sigma_2 &= \nabla_{a_1}^{\text{bas}} \sigma_2 = \Omega_{(\rho, \rho^*)\sigma_2} a_1 + \mathcal{L}_{a_1} \sigma_2 \\ &= \Delta_{(\rho, \rho^*)\sigma_2} \sigma_1 - (0, \mathbf{d}\langle \theta_2, \rho(a_1) \rangle) - (0, \mathcal{L}_{\rho(a_2)} \theta_1) + ([a_1, a_2], \mathcal{L}_{\rho(a_1)} \theta_2) \\ &= \Delta_{(\rho, \rho^*)\sigma_2} \sigma_1 - [\sigma_2, \sigma_1]_D. \end{aligned}$$

□

Lemma 5.16. *Let $(U, K, [\Delta]_{U, K})$ be an \mathcal{LA} -Dirac triple. Then, for all $v \in \Gamma(TM \oplus A^*)$ and $\tau, \sigma \in \Gamma(A \oplus T^*M)$:*

$$(5.16) \quad \langle (\rho, \rho^*) \Delta_v \tau - \llbracket v, (\rho, \rho^*) \tau \rrbracket_{\Delta} - \nabla_{\tau}^{\text{bas}} v, \sigma \rangle = \langle \nabla_{\sigma}^{\text{bas}} v, \tau \rangle.$$

This yields the following corollary.

Corollary 5.17. *Let $(U, K, [\Delta]_{U, K})$ be an \mathcal{LA} -Dirac triple. Then, for all $u \in \Gamma(U)$ and $k \in \Gamma(K)$:*

$$(\rho, \rho^*) \Delta_u k = \llbracket u, (\rho, \rho^*) k \rrbracket_{\Delta} + \nabla_k^{\text{bas}} u.$$

Proof. By Lemma 5.16, we have

$$\langle (\rho, \rho^*) \Delta_u k - \llbracket u, (\rho, \rho^*) k \rrbracket_\Delta - \nabla_k^{\text{bas}} u, \sigma \rangle = \langle \nabla_\sigma^{\text{bas}} u, k \rangle$$

for all $\sigma \in \Gamma(A \oplus T^*M)$. Since $\nabla_\sigma^{\text{bas}}$ preserves $\Gamma(U)$ by Theorem 5.10, this vanishes. \square

Proof of Lemma 5.16. We write $\tau = (b, \theta)$ and $v = (X, \xi)$. Then we have for any $\sigma = (a, \omega) \in \Gamma(A \oplus T^*M)$:

$$\begin{aligned} & \langle (\rho, \rho^*) \Delta_v \tau - \llbracket v, (\rho, \rho^*) \tau \rrbracket_\Delta - \nabla_b^{\text{bas}} v, \sigma \rangle \\ &= \langle \Delta_v \tau - \Omega_v b, (\rho, \rho^*) \sigma \rangle - \langle \mathcal{L}_b v, \sigma \rangle + \langle \llbracket (\rho, \rho^*) \tau, v \rrbracket_\Delta, \sigma \rangle - \langle \text{Skew}_\Delta((\rho, \rho^*) \tau, v), a \rangle \\ &= \langle (0, \mathcal{L}_X \theta + \mathbf{d}(\xi, b)), (\rho, \rho^*) \sigma \rangle - \langle \mathcal{L}_b v, \sigma \rangle + \rho(b) \langle v, \sigma \rangle - \langle v, \Delta_{(\rho, \rho^*) \tau} \sigma \rangle - \langle \text{Skew}_\Delta((\rho, \rho^*) \tau, v), a \rangle \\ &= \langle \mathcal{L}_X \theta, \rho(a) \rangle + \rho(a) \langle \xi, b \rangle + \langle v, \mathcal{L}_b \sigma \rangle - \langle v, \nabla_\sigma^{\text{bas}} \tau - [\sigma, \tau]_D + (0, \mathbf{d} \langle \sigma, (\rho^*, \rho) \tau \rangle) \rangle \\ &\quad - \langle \text{Skew}_\Delta((\rho, \rho^*) \tau, v), a \rangle \quad \text{by Lemma 5.15} \\ &= - \langle v, \nabla_\sigma^{\text{bas}} \tau \rangle + \langle \mathcal{L}_X \theta, \rho(a) \rangle + \rho(a) \langle \xi, b \rangle + \langle v, \mathcal{L}_b \sigma \rangle \\ &\quad + \langle \xi, [a, b] \rangle + \langle \mathcal{L}_{\rho(a)} \theta - \mathcal{L}_{\rho(b)} \omega, X \rangle + X \langle \omega, \rho(b) \rangle - X \langle \sigma, (\rho^*, \rho) \tau \rangle - \langle \text{Skew}_\Delta((\rho, \rho^*) \tau, v), a \rangle \\ &= - \langle v, \nabla_\sigma^{\text{bas}} \tau \rangle - \langle \theta, [X, \rho(a)] \rangle + \rho(a) \langle v, \tau \rangle + \langle v, \mathcal{L}_b \sigma \rangle \\ &\quad + \langle \xi, [a, b] \rangle - \langle \theta, [\rho(a), X] \rangle - \langle \mathcal{L}_{\rho(b)} \omega, X \rangle - \langle \text{Skew}_\Delta((\rho, \rho^*) \tau, v), a \rangle \\ &= - \langle v, \nabla_\sigma^{\text{bas}} \tau \rangle + \rho(a) \langle v, \tau \rangle - \langle \text{Skew}_\Delta((\rho, \rho^*) \tau, v), a \rangle = \langle \nabla_\sigma^{\text{bas}} v, \tau \rangle. \end{aligned}$$

\square

Proof of Theorem 5.14. By (5.9), the equality $U = K^\circ$ and the inclusion $(\rho, \rho^*)(K) \subseteq U$, the bracket $[\cdot, \cdot]_D$ is skew-symmetric on sections of K . Choose $k_1 = (a_1, \theta_1), k_2 \in \Gamma(K)$. Then, by Lemma 5.15, we have

$$[k_1, k_2]_D = \nabla_{a_1}^{\text{bas}} k_2 - \Delta_{(\rho, \rho^*) k_2} k_1$$

Since by Theorem 5.10, the sections $\nabla_{a_1}^{\text{bas}} k_2$ and $\Delta_{(\rho, \rho^*) k_2} k_1$ are elements of $\Gamma(K)$, we find that $[k_1, k_2]_D \in \Gamma(K)$. The Jacobi-identity follows directly from (5.10).

We show next that $(\rho, \rho^*) : K \rightarrow U$ is a Lie algebroid morphism. We have

$$\rho_U \circ (\rho, \rho^*) = \text{pr}_{TM} \circ (\rho, \rho^*) = \rho \circ \text{pr}_A = \rho_K$$

and, for all $k_1, k_2 \in \Gamma(K)$, using Lemma 5.17, Lemma 5.15 and Proposition 5.2:

$$\begin{aligned} \llbracket (\rho, \rho^*) k_1, (\rho, \rho^*) k_2 \rrbracket_\Delta &= (\rho, \rho^*) \Delta_{(\rho, \rho^*) k_1} k_2 - \nabla_{k_2}^{\text{bas}} (\rho, \rho^*) k_1 \\ &= (\rho, \rho^*) (\Delta_{(\rho, \rho^*) k_1} k_2 - \nabla_{k_2}^{\text{bas}} k_1) \\ &= (\rho, \rho^*) ([k_1, k_2]_D). \end{aligned}$$

\square

Example 5.18. (1) In the case of a \mathcal{LA} -Dirac triple as in Example 5.12, we have $K = \{(-\rho_*^* \theta, \theta) \mid \theta \in T^*M\}$ and $U = \{\rho_*(\xi), \xi \mid \xi \in A^*\}$. The Lie algebroid structure on K is just the Lie algebroid defined by the graph of the anchor of the Lie algebroid $(T^*M)_\pi$ with $\pi^\sharp = -\rho \circ \rho_*^* = \rho_* \circ \rho^*$ and the fact that

$$(\rho, \rho^*) : K \rightarrow U$$

recovers the fact that, for a Lie bialgebroid, the map $\rho^* : T^*M \rightarrow A^*$ is a Lie algebroid morphism (see for instance [22], Proposition 12.1.13).

- (2) In the case of a \mathcal{LA} -Dirac triple as in Example 5.13, we have $[(a, \sigma(a)), (b, \sigma(b))]_D = ([a, b], \mathcal{L}_{\rho(a)} \sigma(b) - \mathbf{i}_{\rho(b)} \mathbf{d} \sigma(a)) = ([a, b], \sigma([a, b])) \in \Gamma(K)$ by the results in Example 4.28. Hence, K is a Lie algebroid. The map (ρ, ρ^*) sends $(a, \sigma(a))$ to $(\rho(a), -\sigma^* \rho(a)) \in \Gamma(U)$ since $\rho^* \circ \sigma = -\sigma^* \circ \rho$. The fact that (ρ, ρ^*) is a Lie algebroid morphism also follows from this equality and the fact that ρ is a Lie algebroid morphism.

6. THE MANIN PAIR ASSOCIATED TO AN \mathcal{LA} -DIRAC STRUCTURE

We conclude this paper by showing that the infinitesimal description of an \mathcal{LA} -Dirac structures on Lie algebroids is a Manin pair [7] (C, U) , where $C \rightarrow M$ is a Courant algebroid that is in a particular sense compatible with A and such that the Dirac structure U in C can be seen as a subbundle of $TM \oplus A^*$ with anchor pr_{TM} .

We start by describing the representations up to homotopy describing the two sides of a Dirac algebroid.

6.1. The two VB-Lie algebroid structures on an \mathcal{LA} -Dirac structure. Let $A \rightarrow M$ be a Lie algebroid and D_A a \mathcal{LA} -Dirac structure on A , hence corresponding to a \mathcal{LA} -Dirac triple $(U, K, [\Delta]_{U,K})$.

The Lie algebroid structure $D_{(U,K,[\Delta])} \rightarrow A$ is described by the representation up to homotopy [10] defined by the vector bundle morphism $\text{pr}_A : K \rightarrow A$, the connections

$$(\text{pr}_A \circ \Omega) : \Gamma(U) \times \Gamma(A) \rightarrow \Gamma(A)$$

and

$$\Delta : \Gamma(U) \times \Gamma(K) \rightarrow \Gamma(K),$$

and the curvature $R_\Delta : \Gamma(U^* \otimes U^* \otimes A^* \times K)$.

The Lie algebroid structure of the other side $D_{(U,K,[\Delta])} \rightarrow U$ is described by the complex $(\rho, \rho^*) : K \rightarrow U$ with the basic connections $\nabla^{\text{bas}} : \Gamma(A) \times \Gamma(U) \rightarrow \Gamma(U)$ and $\nabla^{\text{bas}} : \Gamma(A) \times \Gamma(K) \rightarrow \Gamma(K)$ and the basic curvature $R_\Delta^{\text{bas}} \in \Gamma(A^* \otimes A^* \otimes U^* \otimes K)$.

Theorem 6.1. *Let $A \rightarrow M$ be a Lie algebroid and (K, U, Δ) a \mathcal{LA} -Dirac triple. Then:*

(1) *Condition (5) of Theorem 5.10 is equivalent to:*

$$\begin{aligned} \nabla_\sigma^{\text{bas}} \llbracket u, v \rrbracket_\Delta - \llbracket \nabla_\sigma^{\text{bas}} u, v \rrbracket_\Delta - \llbracket u, \nabla_\sigma^{\text{bas}} v \rrbracket_\Delta + \nabla_{\Delta_u \sigma}^{\text{bas}} - \nabla_{\Delta_v \sigma}^{\text{bas}} u = -(\rho, \rho^*) R_\Delta(u, v) \sigma \\ \text{for all } u, v \in \Gamma(U) \text{ and } \sigma \in \Gamma(A \oplus T^*M). \end{aligned}$$

(2) *The equality*

$$\begin{aligned} \Delta_u[\sigma_1, \sigma_2]_D - [\Delta_u \sigma_1, \sigma_2]_D - [\sigma_1, \Delta_u \sigma_2]_D + \Delta_{\nabla_{a_1}^{\text{bas}} u} \sigma_2 - \Delta_{\nabla_{a_2}^{\text{bas}} u} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, \nabla_{a_2}^{\text{bas}} u \rangle) \\ = -R_\Delta^{\text{bas}}(a_1, a_2)u \\ \text{holds for all } \sigma_1, \sigma_2 \in \Gamma(A \oplus T^*M) \text{ with } \text{pr}_A(\sigma_i) =: a_i \text{ and all } u \in \Gamma(U). \end{aligned}$$

The proof of these formulas is quite long, but straightforward. It can be found in Appendix E.

A double Lie algebroid is not only a double vector bundle with VB-algebroid structures on both sides such that the VB-structure maps on each side are Lie algebroid morphisms of the other side. There is an additionnal more complicated criterion that one needs to check, namely the existence of a Lie bialgebroid over the dual of the core of the double vector bundle [21]. The two formulas in the last theorem seem to be part of the compatibility conditions on the two representations up to homotopy, that are necessary for this Lie bialgebroid condition to be satisfied.

6.2. The Courant algebroid associated to an \mathcal{LA} -Dirac triple. Assume that the triple (U, K, Δ) is an \mathcal{LA} -Dirac triple and consider the vector bundle

$$(6.17) \quad C := \frac{U \oplus (A \oplus T^*M)}{\text{graph}(-(\rho, \rho^*)|_K)} \rightarrow M.$$

We will write $u \oplus \sigma$ for the class in C of a pair $(u, \sigma) \in \Gamma(U \oplus (A \oplus T^*M))$. It is easy to check that

$$(6.18) \quad \langle\langle u_1 \oplus \sigma_1, u_2 \oplus \sigma_2 \rangle\rangle_C := \langle u_1, \sigma_2 \rangle + \langle u_2, \sigma_1 \rangle + \langle \sigma_1, (\rho, \rho^*) \sigma_2 \rangle$$

defines a symmetric fiberwise pairing $\langle\langle \cdot, \cdot \rangle\rangle_C$ on C .

Proposition 6.2. *The pairing $\langle\langle\cdot, \cdot\rangle\rangle_C$ is nondegenerate and the vector bundle C is isomorphic to $U \oplus U^*$.*

Proof. The nondegeneracy of $\langle\langle\cdot, \cdot\rangle\rangle_C$ is easy to check.

Define the map $\iota : U \rightarrow C$, $u \mapsto u \oplus 0$ and $\pi : C \rightarrow U^*$, $\pi(u \oplus \sigma)(v) = \langle\langle u \oplus \sigma, v \oplus 0 \rangle\rangle_C$ for all $u \oplus \sigma \in C$, $v \in U$. The map ι is obviously injective. Assume that $\psi(u \oplus \sigma) = 0$. Then $\langle\sigma, v\rangle = 0$ for all $v \in U$. Hence, $\sigma \in K$ and $u \oplus \sigma = (u + (\rho, \rho^*)\sigma) \oplus 0 \in \text{im}(\pi)$. A dimension count yields that π is surjective, and since the sequence

$$0 \rightarrow U \rightarrow C \rightarrow U^* \rightarrow 0$$

is exact, we are done. \square

Set

$$c : C \rightarrow TM,$$

$$c(u \oplus \sigma) = \text{pr}_{TM}(u) + \rho \circ \text{pr}_A(\sigma).$$

Theorem 6.3. *Assume that $(U, K, [\Delta])$ is an \mathcal{LA} -Dirac triple. Then C is a Courant algebroid with anchor c , pairing $\langle\langle\cdot, \cdot\rangle\rangle_C$ and bracket*

$$[\![\cdot, \cdot]\!] : \Gamma(C) \times \Gamma(C) \rightarrow \Gamma(C),$$

$$(6.19) \quad \begin{aligned} & [\![u_1 \oplus \sigma_1, u_2 \oplus \sigma_2]\!] \\ &= ([\![u_1, u_2]\!]_{\Delta} + \nabla_{\sigma_1}^{\text{bas}} u_2 - \nabla_{\sigma_2}^{\text{bas}} u_1) \oplus ([\sigma_1, \sigma_2]_D + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1 + (0, \mathbf{d}\langle\sigma_1, u_2\rangle)). \end{aligned}$$

The map

$$\mathcal{D} = c^* \circ \mathbf{d} : C^\infty(M) \rightarrow \Gamma(C)$$

is given by

$$f \mapsto 0 \oplus (0, \mathbf{d}f).$$

The proof of this theorem can be found in the appendix.

- Remark 6.4.** (1) It is easy to check that the Courant algebroid structure only depends on the (U, K) -equivalence class of Δ .
 (2) This construction has some similarities with the one of matched pairs of Courant algebroids in [11]. It would be interesting to understand the relation between the two constructions.

- Definition 6.5.** (1) A Manin pair over a manifold M is a pair (E, D) of vector bundles over M , such that E is a Courant algebroid and D a Dirac structure in E [7].
 (2) Let $(A \rightarrow M, \rho, [\cdot, \cdot])$ be a Lie algebroid. An A -Manin pair is a Manin pair (C, U) over M , where
 (a) $U \subseteq TM \oplus A^*$ is a subbundle such that $(\rho, \rho^*)(U^\circ) \subseteq U$,
 (b) C is the vector bundle

$$\frac{U \oplus (A \oplus T^*M)}{\text{graph}(-(\rho, \rho^*)|_{U^\circ})} \rightarrow M$$

endowed with the anchor $c = \text{pr}_{TM} \oplus \rho \circ \text{pr}_A$ and the pairing $\langle\langle\cdot, \cdot\rangle\rangle_C$.

- (c) The bracket of C satisfies $[\![0 \oplus \sigma_1, 0 \oplus \sigma_2]\!] = 0 \oplus [\sigma_1, \sigma_2]_D$ for all $\sigma_1, \sigma_2 \in \Gamma(A \oplus T^*M)$.

The final theorem and second main result of this paper is the following:

Theorem 6.6. *Let A be a Lie algebroid over a manifold M . Then there is a one-one correspondence between \mathcal{LA} -Dirac structures on A and A -Manin pairs.*

Proof. We have already seen that \mathcal{LA} -Dirac structures on a Lie algebroid A are in bijection with \mathcal{LA} -Dirac triples on A . We show here that there is a one-one correspondence between \mathcal{LA} -Dirac triples on A and A -Manin pairs. We have seen in Theorem 6.3 how to associate an A -Manin pair to an \mathcal{LA} -Dirac triple on A .

Conversely, choose an A -Manin pair (C, U) and set $K := U^\circ \subseteq A \oplus T^*M$. Then $U \simeq U \oplus 0$ is a Dirac structure in C and there is hence an induced Dorfman connection

$$\Delta^U : \Gamma(U) \times \Gamma(C/U) \rightarrow \Gamma(C/U).$$

It is easy to verify that the map $C/U \rightarrow (A \oplus T^*M)/U^\circ$ sending $\overline{u \oplus \sigma} \in C/U$ to $\bar{\sigma} \in (A \oplus T^*M)/U^\circ$ is an isomorphism of vector bundles. Using the Leibniz equality in both arguments, extend the Lie algebroid bracket of U to a dull algebroid structure on $TM \oplus A^*$ with anchor pr_{TM} . It is easy to see that the corresponding $TM \oplus A^*$ -Dorfman connection Δ on $A \oplus T^*M$ satisfies $\Delta_u k \in \Gamma(K)$ for all $k \in \Gamma(K)$ and $u \in \Gamma(U)$, and that the induced U -Dorfman connection on the quotient $(A \oplus T^*M)/K$ is equal to Δ^U . Furthermore, for two dull extensions of $[\cdot, \cdot]_U$, we find that the corresponding Dorfman connections are (U, K) -equivalent. Hence, we can write $\Delta^U = [\Delta]$. We check that (U, K, Δ^U) is an \mathcal{LA} -Dirac triple.

For this, we only check that the Courant bracket of C is defined as in (6.19). The proof of Theorem 6.3 shows that all the conditions in Theorem 5.10 are then satisfied.

Choose $\tau = (a, \theta) \in \Gamma(A \oplus T^*M)$, $\sigma = (b, \omega) \in \Gamma(A \oplus T^*M)$ and $u = (X, \xi) \in \Gamma(U)$. We want to compute $v = v(\tau, u) \in \Gamma(U)$ such that $\llbracket u \oplus 0, 0 \oplus \tau \rrbracket = v \oplus \Delta_u \tau$. Note first that

$$\llbracket u \oplus 0, 0 \oplus \tau \rrbracket + \llbracket 0 \oplus \tau, u \oplus 0 \rrbracket = \mathcal{D}\langle\langle u \oplus 0, 0 \oplus \tau \rangle\rangle_C = \mathcal{D}\langle\tau, u\rangle.$$

The map $\mathcal{D} : C^\infty(M) \rightarrow \Gamma(C)$ is given by

$$\langle\langle u \oplus \sigma, \mathcal{D}\varphi \rangle\rangle_C = (\text{pr}_{TM}(u) + \rho \circ \text{pr}_A(\sigma))\varphi$$

for all $u \oplus \sigma \in \Gamma(C)$, i.e. $\mathcal{D}\varphi = 0 \oplus (0, \mathbf{d}\varphi)$. Then, by the Leibniz property of the Courant algebroid bracket on C , we find

$$\begin{aligned} \rho(a)\langle u, \sigma \rangle &= c(0 \oplus \tau)\langle\langle u \oplus 0, 0 \oplus \sigma \rangle\rangle_C \\ &= \langle\langle \llbracket 0 \oplus \tau, u \oplus 0 \rrbracket, 0 \oplus \sigma \rangle\rangle_C + \langle\langle u \oplus 0, \llbracket 0 \oplus \tau, 0 \oplus \sigma \rrbracket \rangle\rangle_C \\ &= \langle\langle (-v) \oplus (-\Delta_u \tau + (0, \mathbf{d}\langle\tau, u\rangle)), 0 \oplus \sigma \rangle\rangle_C + \langle\langle u \oplus 0, 0 \oplus (\mathcal{L}_a \sigma + (0, -\mathbf{i}_{\rho(b)} \mathbf{d}\theta)) \rangle\rangle_C \\ &= -\langle v, \sigma \rangle - \langle (\rho, \rho^*) \Omega_u a, \sigma \rangle - \langle \mathcal{L}_X \theta, \rho(b) \rangle + \rho(b) \langle \theta, X \rangle + \langle u, \mathcal{L}_a \sigma \rangle - \mathbf{d}\theta(\rho(b), X) \end{aligned}$$

This leads to

$$-\langle v, \sigma \rangle = \langle (\rho, \rho^*) \Omega_u a, \sigma \rangle + \langle \mathcal{L}_a u, \sigma \rangle$$

and, since σ was arbitrary, we have shown that

$$\llbracket u \oplus 0, 0 \oplus \tau \rrbracket = (-\nabla_\tau^{\text{bas}} u) \oplus \Delta_u \tau.$$

□

Example 6.7. Consider an \mathcal{LA} -Dirac triple as in Example 5.12. The vector bundle morphisms

$$\Psi : C \rightarrow A \oplus A^*, \quad \Psi((\rho_* \xi, \xi) \oplus (a, \theta)) = (a + \rho_*^* \theta, \xi + \rho^* \theta)$$

and

$$\Phi : A \oplus A^* \rightarrow C, \quad \Phi(a, \xi) = (\rho_* \xi, \xi) \oplus (a, 0)$$

are well-defined and inverse to each other. A straightforward computation using the considerations in Examples 4.17 and 5.12 shows that C is isomorphic to the Courant algebroid structure on $A \oplus A^*$ induced by the Lie bialgebroid (A, A^*) [20]⁴

⁴In [20], the authors work with the definition of Courant algebroids with antisymmetric brackets. Here, we get the corresponding Courant algebroid as we chose to define them.

Example 6.8. Consider now an \mathcal{LA} -Dirac triple as in Example 5.13. Again, we find that the vector bundle morphisms

$$\Pi : C \rightarrow TM \oplus T^*M, \quad \Pi((X, -\sigma^*X) \oplus (a, \theta)) = (X + \rho(a), \theta + \sigma(a))$$

and

$$\Theta : TM \oplus T^*M \rightarrow C, \quad \Theta(X, \theta) = (X, -\sigma^*X) \oplus (0, \theta)$$

are well-defined and inverse to each other. Here, one gets immediately that C and the standard Courant algebroid $TM \oplus T^*M$ are isomorphic via these maps.

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APPENDIX A. LINEAR ALMOST POISSON STRUCTURES ON E AND SPLITTINGS OF $TE \oplus T^*E$

Consider here a skew-symmetric dull algebroid $(A, \rho, [\cdot, \cdot])$. This is equivalent to a linear almost Poisson bracket on the vector bundle $A^* \rightarrow M$, i.e. a skew-symmetric bracket $\{\cdot, \cdot\} : C^\infty(A^*) \times C^\infty(A^*) \rightarrow C^\infty(A^*)$ such that

- (1) $\{\cdot, \cdot\}$ satisfies the Leibniz identity,
- (2) $\{\ell_a, \ell_b\} = \ell_{[a, b]}$ is again linear for two sections $a, b \in \Gamma(A)$ and
- (3) $\{\ell_a, q_{A^*}^* \varphi\} = q_{A^*}^*(\rho(a)(\varphi))$ is again a pullback for all $a \in \Gamma(A)$ and $\varphi \in C^\infty(M)$.

Let $\pi_A \in \mathfrak{X}^2(A^*)$ be the bivector field associated to this almost Poisson structure. We will describe the vector bundle morphism $\pi_A^\sharp : T^*A^* \rightarrow TA^*$, $\mathbf{d}F \mapsto \{F, \cdot\}$, $F \in C^\infty(A^*)$, associated to it.

We compute the vector fields $\pi_A^\sharp(\mathbf{d}\ell_a)$ and $\pi_A^\sharp(\theta^\dagger)$ for all $a \in \Gamma(A)$ and $\theta \in \Omega^1(M)$. Since $\pi_A^\sharp(q_{A^*}^* \mathbf{d}\varphi)(q_{A^*}^* \psi) = \pi_A^\sharp(\mathbf{d}q_{A^*}^* \varphi)(q_{A^*}^* \psi) = 0$ and $\pi_A^\sharp(\mathbf{d}q_{A^*}^* \varphi)(\ell_a) = -q_{A^*}^*(\rho(a)(\varphi))$ for all $\varphi, \psi \in C^\infty(M)$ and $a \in \Gamma(A)$, we find $\pi_A^\sharp(q_{A^*}^* \mathbf{d}\varphi) = -(\rho^*(\mathbf{d}\varphi))^\dagger$ for all $\varphi \in C^\infty(M)$ and consequently $\pi_A^\sharp(q_{A^*}^* \theta) = -(\rho^* \theta)^\dagger$ for all $\theta \in \Omega^1(M)$.

In the same manner, we have

$$\begin{aligned} \pi_A^\sharp(\mathbf{d}\ell_a)(\ell_b) &= \ell_{[a, b]} \quad \text{and} \\ \pi_A^\sharp(\mathbf{d}\ell_a)(q_{A^*}^* \varphi) &= q_{A^*}^*(\rho(a)(\varphi)) \end{aligned}$$

for $a, b \in \Gamma(A)$ and $\varphi \in C^\infty(M)$. Consider the vector field $\widetilde{\rho(a)} \in \mathfrak{X}(A^*)$ that is defined by

$$\widetilde{\rho(a)}(\xi_m) = T_m \xi(\rho(a)(m)) - \left. \frac{d}{dt} \right|_{t=0} \xi_m + t \mathcal{L}_a \xi(m)$$

for all $\xi_m \in A^*$ and any section $\xi \in \Gamma(A^*)$ such that $\xi(m) = \xi_m$. Since

$$\begin{aligned} \widetilde{\rho(a)}(\xi_m)(\ell_b) &= \rho(a)(m) \langle \xi, b \rangle - \langle \mathcal{L}_a \xi, b \rangle(m) \\ &= \langle \xi, [a, b] \rangle(m) = \ell_{[a, b]}(\xi_m) \quad \text{and} \\ \widetilde{\rho(a)}(\xi_m)(q_{A^*}^* \varphi) &= \rho(a)(m)(\varphi), \end{aligned}$$

we have shown the equality

$$\pi_A^\sharp(\widetilde{a}) = \widetilde{\rho(a)}.$$

APPENDIX B. THE CANONICAL SYMPLECTIC FORM AND TM -CONNECTIONS

We consider here the vector bundle $c_M : T^*M \rightarrow M$ over a manifold M . Recall that there is a canonical 1-form $\theta_{\text{can}} \in \Omega^1(T^*M)$, given by

$$\langle \theta_{\text{can}}(\alpha_m), x_{\alpha_m} \rangle = \langle \alpha_m, T_{\alpha_m} c_M(x_{\alpha_m}) \rangle$$

for all $\alpha_m \in T^*M$ and $x_{\alpha_m} \in T_{\alpha_m}(T^*M)$. The canonical 2-form ω_{can} is defined by

$$\omega_{\text{can}} = -\mathbf{d}\theta_{\text{can}}$$

and $(T^*M, \omega_{\text{can}})$ is a symplectic manifold.

Choose any connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ and consider the linear section $\tilde{X} = \widehat{\nabla}^* \in \mathfrak{X}(T^*M)$ over $X \in \mathfrak{X}(M)$ defined by the dual connection $\nabla^* : \mathfrak{X}(M) \times \Omega^1(M) \rightarrow \Omega^1(M)$ as in Section 2.3.3 and the core sections α^\dagger for $\alpha \in \Omega^1(M)$. We have then, by definition of the horizontal and core sections:

$$\begin{aligned} \langle \theta_{\text{can}}(\alpha_m), \tilde{X}(\alpha_m) \rangle &= \langle \alpha_m, X(m) \rangle \\ \langle \theta_{\text{can}}(\alpha_m), \beta^\dagger(\alpha_m) \rangle &= 0 \end{aligned}$$

for all $\alpha_m \in T^*M$, $X \in \mathfrak{X}(M)$ and $\beta \in \Omega^1(M)$. This shows in particular the equality $\langle \theta_{\text{can}}, \tilde{X} \rangle = \ell_X$.

We have hence, for all $\beta, \gamma \in \Omega^1(M)$, $X, Y \in \mathfrak{X}(M)$:

$$\begin{aligned} \omega_{\text{can}}(\tilde{X}, \tilde{Y}) &= \tilde{X}(\theta_{\text{can}}(\tilde{Y})) - \tilde{Y}(\theta_{\text{can}}(\tilde{X})) - \theta_{\text{can}}([\tilde{X}, \tilde{Y}]) \\ &= \tilde{X}(\ell_Y) - \tilde{Y}(\ell_X) - \theta_{\text{can}}(\widetilde{[X, Y]}) - R_{\nabla^*}(X, Y)^\uparrow \\ &= \ell_{\nabla_X Y} - \ell_{\nabla_Y X} - \ell_{[X, Y]} = \ell_{\nabla_X Y - \nabla_Y X - [X, Y]}, \\ \omega_{\text{can}}(\tilde{X}, \alpha^\uparrow) &= \tilde{X}(0) - \alpha^\uparrow(\ell_X) - \theta_{\text{can}}([\tilde{X}, \alpha^\uparrow]) = -c_M^* \langle \alpha, X \rangle \\ \omega_{\text{can}}(\alpha^\uparrow, \beta^\uparrow) &= 0. \end{aligned}$$

Hence, the one-form $\omega_{\text{can}}^b(\tilde{X})$ is given by

$$\omega_{\text{can}}^b(\tilde{X}) = -\mathbf{d}\ell_X + \langle \cdot, \text{Tor}_{\nabla}(X, \cdot) + \nabla_X \cdot \rangle^\uparrow,$$

where $\langle \cdot, \text{Tor}_{\nabla}(X, \cdot) + \nabla_X \cdot \rangle$ is seen as a section of $\text{Hom}(T^*M, T^*M)$.

Consider now a vector bundle $E \rightarrow M$ endowed with a vector bundle morphism $\sigma : E \rightarrow T^*M$ over the identity and a connection $\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$. The one-form $\sigma^* \theta_{\text{can}} \in \Omega^1(E)$ can, here also, be characterized as follows

$$\begin{aligned} \langle (\sigma^* \theta_{\text{can}})(e_m), \tilde{X}(e_m) \rangle &= \langle \theta_{\sigma(e_m)}, T_{e_m} \sigma(\tilde{X}(e_m)) \rangle \\ &= \langle \sigma(e_m), X(m) \rangle \\ \langle (\sigma^* \theta_{\text{can}})(e_m), f^\uparrow(e_m) \rangle &= \langle \theta_{\sigma(e_m)}, \sigma(f)^\uparrow(e_m) \rangle = 0 \end{aligned}$$

for all $\alpha_m \in T^*M$, $X \in \mathfrak{X}(M)$ and $\beta \in \Omega^1(M)$. This shows in particular the equality $\langle \sigma^* \theta_{\text{can}}, \tilde{X} \rangle = \ell_{\sigma^* X}$.

We have hence, for all $e, f \in \Omega^1(M)$, $X, Y \in \mathfrak{X}(M)$:

$$\begin{aligned} \sigma^* \omega_{\text{can}}(\tilde{X}, \tilde{Y}) &= \tilde{X}((\sigma^* \theta_{\text{can}})(\tilde{Y})) - \tilde{Y}((\sigma^* \theta_{\text{can}})(\tilde{X})) - (\sigma^* \theta_{\text{can}})([\tilde{X}, \tilde{Y}]) \\ &= \tilde{X}(\ell_{\sigma^* Y}) - \tilde{Y}(\ell_{\sigma^* X}) - (\sigma^* \theta_{\text{can}})(\widetilde{[X, Y]}) - R_{\nabla}(X, Y)^\uparrow \\ &= \ell_{\nabla_X^* (\sigma^* Y)} - \ell_{\nabla_Y^* (\sigma^* X)} - \ell_{\sigma^* [X, Y]} = \ell_{\nabla_X^* (\sigma^* Y) - \nabla_Y^* (\sigma^* X) - \sigma^* [X, Y]}, \\ \sigma^* \omega_{\text{can}}(\tilde{X}, e^\uparrow) &= \tilde{X}(0) - e^\uparrow(\ell_{\sigma^* X}) - \sigma^* \theta_{\text{can}}([\tilde{X}, e^\uparrow]) = -q_E^* \langle \sigma(e), X \rangle \\ \sigma^* \omega_{\text{can}}(e^\uparrow, f^\uparrow) &= 0. \end{aligned}$$

Hence, the one-forms $(\sigma^* \omega_{\text{can}})^b(\tilde{X})$ and $(\sigma^* \omega_{\text{can}})^b(e^\uparrow) \in \Omega^1(E)$ are given by

$$(\sigma^* \omega_{\text{can}})^b(\tilde{X}) = \mathbf{d}\ell_{-\sigma^* X} + (\sigma(\nabla_X \cdot) - \mathcal{L}_X(\sigma(\cdot)))^\uparrow,$$

where $\sigma(\nabla_X \cdot) - \mathcal{L}_X(\sigma(\cdot))$ is seen as a section of $\text{Hom}(E, T^*M)$, and

$$(\sigma^* \omega_{\text{can}})^b(e^\uparrow) = q_E^*(\sigma(e)).$$

APPENDIX C. PROOF OF THEOREM 4.13

We prove in this section Theorem 4.13. For simplicity, given a Dorfman connection

$$\Delta : \Gamma(TM \oplus E^*) \times \Gamma(E \oplus T^*M) \rightarrow \Gamma(E \oplus T^*M),$$

we will write $\tilde{X} = \text{pr}_{TE}(\widetilde{X, \xi})$ and $\tilde{\xi} = \text{pr}_{T^*E}(\widetilde{X, \xi})$.

We prove first a few technical lemmas.

Lemma C.1. *The Dorfman connection can be written*

$$\Delta_{(X, \xi)}(e, \theta) = \Delta_{(X, \xi)}(e, 0) + (0, \mathcal{L}_X \theta)$$

for all $(X, \xi) \in \Gamma(TM \oplus E^*)$ and $(e, \theta) \in \Gamma(E \oplus T^*M)$.

Proof. This is proved in Remark 4.7. □

Lemma C.2. Choose $(X, \xi), (Y, \eta), (Z, \chi) \in \Gamma(TM \oplus E^*)$ and $e \in \Gamma(E)$. Then

- (1) $\text{pr}_E \Delta_{(Y, \eta)} (\text{pr}_E \Delta_{(X, \xi)}(e, 0), 0) = \text{pr}_E (\Delta_{(Y, \eta)} \Delta_{(X, \xi)}(e, 0)),$
- (2) $\langle \text{pr}_E \Delta_{(X, \xi)}(e, 0), 0, \llbracket (Y, \eta), (Z, \chi) \rrbracket_\Delta \rangle + \langle \text{pr}_{T^*M} \Delta_{(X, \xi)}(e, 0), [Y, Z] \rangle$
 $= \langle \Delta_{(X, \xi)}(e, 0), \llbracket (Y, \eta), (Z, \chi) \rrbracket_\Delta \rangle.$

Proof. The second formula is immediate. For the first one, choose $\chi \in \Gamma(E^*)$. Then

$$\begin{aligned} \langle \chi, \text{pr}_E \Delta_{(Y, \eta)} (\text{pr}_E \Delta_{(X, \xi)}(e, 0), 0) \rangle &= \langle (0, \chi), \Delta_{(Y, \eta)} (\text{pr}_E \Delta_{(X, \xi)}(e, 0), 0) \rangle \\ &= Y \langle (0, \chi), (\text{pr}_E \Delta_{(X, \xi)}(e, 0), 0) \rangle \\ &\quad - \langle \llbracket (Y, \eta), (0, \chi) \rrbracket_\Delta, (\text{pr}_E \Delta_{(X, \xi)}(e, 0), 0) \rangle \\ &= Y \langle (0, \chi), \Delta_{(X, \xi)}(e, 0) \rangle - \langle \llbracket (Y, \eta), (0, \chi) \rrbracket_\Delta, \Delta_{(X, \xi)}(e, 0) \rangle \\ &= \langle (0, \chi), \Delta_{(Y, \eta)} \Delta_{(X, \xi)}(e, 0) \rangle \\ &= \langle \chi, \text{pr}_E \Delta_{(Y, \eta)} \Delta_{(X, \xi)}(e, 0) \rangle. \end{aligned}$$

In the third equality, we have used the fact that $\text{pr}_{TM} \llbracket (Y, \eta), (0, \chi) \rrbracket_\Delta = 0$. Since χ was arbitrary, we are done. \square

Lemma C.3. Choose $(X, \xi) \in \Gamma(TM \oplus E^*)$.

- (1) For any section $\eta \in \Gamma(E^*)$, the function $\tilde{X}(\ell_\eta)$ is again linear and given by $\tilde{X}(\ell_\eta) = \ell_{\psi^{(X, \xi), \eta}}$, where $\psi^{(X, \xi), \eta} \in \Gamma(E^*)$ is defined by

$$\langle \psi^{(X, \xi), \eta}, e \rangle = X \langle \eta, e \rangle - \langle \eta, \text{pr}_E \Delta_{(X, \xi)}(e, 0) \rangle.$$

- (2) For any $e \in \Gamma(E)$, we have $\langle \tilde{\xi}, e^\uparrow \rangle = q_E^* \langle \xi, e \rangle$.
- (3) For any $e \in \Gamma(E)$, the Lie derivative $\mathcal{L}_{e^\uparrow} \tilde{\xi}$ equals $q_E^* (\mathbf{d} \langle \xi, e \rangle - \text{pr}_{T^*M} \Delta_{(X, \xi)}(e, 0))$.
- (4) For any $e \in \Gamma(E)$, we have $[\tilde{X}, e^\uparrow] = (\text{pr}_E \Delta_{(X, \xi)}(e, 0))^\uparrow$.

Proof. (1) The function $\psi^{(X, \xi), \eta} : E \rightarrow \mathbb{R}$ is easily seen to be $C^\infty(M)$ -linear and hence a section of E^* . A straightforward computation using the definition of \tilde{X} and the properties of Δ completes the proof.

- (2) This is immediate by the definition of $\tilde{\xi}$.
- (3) For any $f \in \Gamma(E)$, we have

$$\langle \mathcal{L}_{e^\uparrow} \tilde{\xi}, f^\uparrow \rangle = e^\uparrow (\langle \tilde{\xi}, f^\uparrow \rangle) - \langle \tilde{\xi}, [e^\uparrow, f^\uparrow] \rangle = e^\uparrow (q_E^* \langle \xi, f \rangle) - \langle \tilde{\xi}, 0 \rangle = 0.$$

This shows that $\mathcal{L}_{e^\uparrow} \tilde{\xi}$ is vertical, i.e. the pullback under q_E of a 1-form on M . Thus, we just need to compute

$$\langle (\mathcal{L}_{e^\uparrow} \tilde{\xi})(f(m)), T_m f x_m \rangle$$

for $f \in \Gamma(E)$ and $x_m \in TM$. But we have

$$\begin{aligned} \langle (\mathcal{L}_{e^\uparrow} \tilde{\xi})(f(m)), T_m f x_m \rangle &= \left. \frac{d}{dt} \right|_{t=0} \langle \tilde{\xi}(f(m) + te(m)), T_{f(m)} \phi_t^{e^\uparrow}(T_m f x_m) \rangle \\ &= \left. \frac{d}{dt} \right|_{t=0} \langle \tilde{\xi}(f(m) + te(m)), T_m(f + te)x_m \rangle \\ &= \left. \frac{d}{dt} \right|_{t=0} x_m \langle \xi, f + te \rangle - \langle \text{pr}_{T^*M} \Delta_{(X, \xi)}(f + te, 0), x_m \rangle \\ &= x_m \langle \xi, e \rangle - \langle \text{pr}_{T^*M} \Delta_{(X, \xi)}(e, 0), x_m \rangle. \end{aligned}$$

- (4) Since $\tilde{X} \sim_{q_E} X$ and $e^\uparrow \sim_{q_E} 0$, we have $[\tilde{X}, e^\uparrow] \sim_{q_E} 0$ and the Lie bracket is a vertical vector field. We compute thus just $[\tilde{X}, e^\uparrow](\ell_\eta)$ for sections $\eta \in \Gamma(E^*)$. We have

$$\begin{aligned} [\tilde{X}, e^\uparrow](\ell_\eta) &= \tilde{X}(e^\uparrow(\ell_\eta)) - e^\uparrow(\tilde{X}(\ell_\eta)) = \tilde{X}(q_E^* \langle \eta, e \rangle) - e^\uparrow(\ell_{\psi(X, \xi), \eta}) \\ &= q_E^* \left(X \langle \eta, e \rangle - \langle \psi^{(X, \xi), \eta}, e \rangle \right) \\ &= q_E^* \langle \eta, \text{pr}_E \Delta_{(X, \xi)}(e, 0) \rangle = (\text{pr}_E \Delta_{(X, \xi)}(e, 0))^\uparrow(\ell_\eta). \end{aligned}$$

□

Since \tilde{X} is linear over X , the flow $\phi_t^{\tilde{X}}$ is a vector bundle morphism $E \rightarrow E$ over $\phi_t^X : M \rightarrow M$, for any $t \in \mathbb{R}$ where this is defined. Hence, for any section $e \in \Gamma(E)$, we can define a new section $\psi_t^X(e) \in \Gamma(E)$ by

$$\psi_t^X(e) = \phi_{-t}^{\tilde{X}} \circ e \circ \phi_t^X.$$

Lemma C.4. *The time derivative of ψ_t^X satisfies*

$$\left. \frac{d}{dt} \right|_{t=0} \psi_t^X(e) = \text{pr}_E \Delta_{(X, \xi)}(e, 0).$$

Proof. The curve $c : t \mapsto \psi_t^X(e)(m)$ is a curve in E with $c(0) = e_m$ and satisfying $q_E \circ c = m$. Hence, the derivative $\dot{c}(0)$ is a vertical vector over e_m . Since $\phi_t^{\tilde{X}}$ is linear, we have

$$((\phi_t^{\tilde{X}})^* e^\uparrow)(f_m) = \left. \frac{d}{ds} \right|_{s=0} \phi_{-t}^{\tilde{X}}(\phi_t^{\tilde{X}}(f_m) + s e(\phi_t^X(m))) = \left. \frac{d}{ds} \right|_{s=0} f_m + s \phi_{-t}^{\tilde{X}}(e(\phi_t^X(m)))$$

for $f_m \in E$. Thus, we get for any $\eta \in \Gamma(E^*)$:

$$\begin{aligned} [\tilde{X}, e^\uparrow](\ell_\eta)(f_m) &= \left. \frac{d}{dt} \right|_{t=0} \langle \mathbf{d}_{f_m} \ell_\eta, ((\phi_t^{\tilde{X}})^* e^\uparrow) \rangle \\ &= \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} \langle \eta(m), f_m + s \phi_{-t}^{\tilde{X}}(e(\phi_t^X(m))) \rangle \\ &= \left. \frac{d}{ds} \right|_{s=0} \left. \frac{d}{dt} \right|_{t=0} \langle \eta(m), f_m + s \phi_{-t}^{\tilde{X}}(e(\phi_t^X(m))) \rangle \\ &= \left. \frac{d}{ds} \right|_{s=0} s \langle \eta(m), \left. \frac{d}{dt} \right|_{t=0} \psi_t^X(e)(m) \rangle = \langle \eta(m), \left. \frac{d}{dt} \right|_{t=0} \psi_t^X(e)(m) \rangle. \end{aligned}$$

This shows that

$$[\tilde{X}, e^\uparrow] = \left(\left. \frac{d}{dt} \right|_{t=0} \psi_t^X(e) \right)^\uparrow.$$

By (4) of Lemma C.3, we are done. □

Now we can prove Theorem 4.13.

Proof of Theorem 4.13. (1) The first equality is easy to check: for the tangent part, we use the fact that the flows of the vertical vector fields commute. For the cotangent part, note that since $e^\uparrow \sim_{q_E} 0$ and $f^\uparrow \sim_{q_E} 0$ and $\theta^\uparrow = q_E^* \theta$, $\omega^\uparrow = q_E^* \omega$, we get immediately $\mathcal{L}_{e^\uparrow} \omega^\uparrow - \mathbf{i}_{f^\uparrow} \mathbf{d} \theta^\uparrow = q_E^* (\mathcal{L}_0 \omega - \mathbf{i}_0 \mathbf{d} \theta) = 0$.

(2) For the second equality, we know by Lemma C.3 that $[\tilde{X}, e^\uparrow] = (\text{pr}_E \Delta_{(X, \xi)}(e, 0))^\uparrow$.

We compute the cotangent part of the Courant-Dorfman bracket. Using Lemma C.3, we have

$$\begin{aligned} \mathcal{L}_{\tilde{X}} \theta^\uparrow - \mathbf{i}_{e^\uparrow} \mathbf{d}\tilde{\xi} &= \mathcal{L}_{\tilde{X}} \theta^\uparrow - \mathcal{L}_{e^\uparrow} \tilde{\xi} + \mathbf{d}\langle \tilde{\xi}, e^\uparrow \rangle \\ &= \mathcal{L}_{\tilde{X}} q_E^* \theta - q_E^* (\mathbf{d}\langle \xi, e \rangle - \text{pr}_{T^*M} \Delta_{(X, \xi)}(e, 0)) + \mathbf{d}q_E^* \langle \xi, e \rangle \\ &= q_E^* (\mathcal{L}_X \theta + \text{pr}_{T^*M} \Delta_{(X, \xi)}(e, 0)). \end{aligned}$$

This leads to

$$\left[\widetilde{(X, \xi)}, (e, \theta)^\uparrow \right] = (0, \mathcal{L}_{X, \theta})^\uparrow + \Delta_{(X, \xi)}(e, 0)^\uparrow.$$

By the second formula in Lemma C.1, we are done.

(3) Choose a section (f, θ) of $E \oplus T^*M$. Then:

$$\left\langle \mathcal{L}_{\tilde{X}} \tilde{\eta} - \mathbf{i}_{\tilde{Y}} \mathbf{d}\tilde{\xi}, f^\uparrow \right\rangle = \tilde{X} \langle \tilde{\eta}, f^\uparrow \rangle - \left\langle \tilde{\eta}, [\tilde{X}, f^\uparrow] \right\rangle - \tilde{Y} \langle \tilde{\xi}, f^\uparrow \rangle + f^\uparrow \langle \tilde{\xi}, \tilde{Y} \rangle + \left\langle \tilde{\xi}, [\tilde{Y}, f^\uparrow] \right\rangle$$

We have

- (a) $\langle \tilde{\eta}, f^\uparrow \rangle = q_E^* \langle \eta, f \rangle$ by Lemma C.3, and consequently $\tilde{X} \langle \tilde{\eta}, f^\uparrow \rangle = q_E^* (X \langle \eta, f \rangle)$.
- (b) $[\tilde{X}, f^\uparrow] = (\text{pr}_E \Delta_{(X, \xi)}(f, 0))^\uparrow$ by (2) and $\left\langle \tilde{\eta}, [\tilde{X}, f^\uparrow] \right\rangle = q_E^* \langle \eta, \text{pr}_E \Delta_{(X, \xi)}(f, 0) \rangle$ follows.
- (c) $\langle \tilde{\xi}, \tilde{Y} \rangle(e_m) = Y(m) \langle \xi, e \rangle - \langle \xi, \text{pr}_E \Delta_{(Y, \eta)}(e, 0) \rangle(m) - \langle Y, \text{pr}_{T^*M} \Delta_{(X, \xi)}(e, 0) \rangle(m)$, which defines a linear function on E . This yields $f^\uparrow \langle \tilde{\xi}, \tilde{Y} \rangle = q_E^* (Y \langle \xi, f \rangle - \langle \xi, \text{pr}_E \Delta_{(Y, \eta)}(f, 0) \rangle - \langle Y, \text{pr}_{T^*M} \Delta_{(X, \xi)}(f, 0) \rangle)$.

Thus, we get

$$\begin{aligned} \left\langle \mathcal{L}_{\tilde{X}} \tilde{\eta} - \mathbf{i}_{\tilde{Y}} \mathbf{d}\tilde{\xi}, f^\uparrow \right\rangle &= q_E^* (X \langle \eta, f \rangle - \langle \eta, \text{pr}_E \Delta_{(X, \xi)}(f, 0) \rangle - Y \langle \xi, f \rangle \\ &\quad + Y \langle \xi, f \rangle - \langle \xi, \text{pr}_E \Delta_{(Y, \eta)}(f, 0) \rangle \\ &\quad - \langle Y, \text{pr}_{T^*M} \Delta_{(X, \xi)}(f, 0) \rangle + \langle \xi, \text{pr}_E \Delta_{(Y, \eta)}(f, 0) \rangle) \\ &= q_E^* (X \langle \eta, f \rangle - \langle (Y, \eta), \Delta_{(X, \xi)}(f, 0) \rangle) \\ &= q_E^* (\llbracket (X, \xi), (Y, \eta) \rrbracket_\Delta, (f, 0)). \end{aligned}$$

This leads to

$$\begin{aligned} \left\langle \left[\widetilde{(X, \xi)}, \widetilde{(Y, \eta)} \right], (f, \theta)^\uparrow \right\rangle &= q_E^* (\langle \theta, [X, Y] \rangle + \langle \llbracket (X, \xi), (Y, \eta) \rrbracket_\Delta, (f, 0) \rangle) \\ &= q_E^* \langle \llbracket (X, \xi), (Y, \eta) \rrbracket_\Delta, (f, \theta) \rangle, \end{aligned}$$

which shows that

$$\left[\widetilde{(X, \xi)}, \widetilde{(Y, \eta)} \right] (e_m) = (T_m e[X, Y](m), \mathbf{d}_{e_m} \ell_{\text{pr}_{E^*} \llbracket (X, \xi), (Y, \eta) \rrbracket_\Delta}) + (g, \omega)^\uparrow (e_m),$$

for some $g \in \Gamma(E)$, $\omega \in \Omega^1(M)$.

As in the proof of Proposition 4.8, we know that for any $(Z, \chi) \in \Gamma(TM \oplus E^*)$, we have

$$\begin{aligned} \langle (g, \omega), (Z, \chi) \rangle(m) &= \left\langle \left[\widetilde{(X, \xi)}, \widetilde{(Y, \eta)} \right] (e_m), (T_m eZ(m), \mathbf{d}_{e_m} \ell_\chi) \right\rangle \\ &\quad - Z(m) \langle \llbracket (X, \xi), (Y, \eta) \rrbracket_\Delta, (e, 0) \rangle - [X, Y] \langle \chi, e \rangle. \end{aligned}$$

We start by computing $[\tilde{X}, \tilde{Y}](\ell_\chi)$. Using Lemma C.3, we have $\tilde{Y}(l_\chi) = \ell_{\psi^{(Y,\eta),\chi}}$ and hence

$$\begin{aligned} \tilde{X}\tilde{Y}(\ell_\chi)(e_m) &= \tilde{X}(\ell_{\psi^{(Y,\eta),\chi}})(e_m) \\ &= \left(X \langle \psi^{(Y,\eta),\chi}, e \rangle - \langle \psi^{(Y,\eta),\chi}, \text{pr}_E \Delta_{(X,\xi)}(e, 0) \rangle \right) (m) \\ &= (XY \langle \chi, e \rangle - X \langle \chi, \text{pr}_E \Delta_{(Y,\eta)}(e, 0) \rangle \\ &\quad - Y \langle \chi, \text{pr}_E \Delta_{(X,\xi)}(e, 0) \rangle + \langle \chi, \text{pr}_E \Delta_{(Y,\eta)}(\text{pr}_E \Delta_{(X,\xi)}(e, 0), 0) \rangle) (m). \end{aligned}$$

Using Lemma C.2, we get

$$[\tilde{X}, \tilde{Y}](\ell_\chi)(e_m) = ([X, Y] \langle \chi, e \rangle + \langle \chi, \text{pr}_E (\Delta_{(Y,\eta)} \Delta_{(X,\xi)}(e, 0) - \Delta_{(X,\xi)} \Delta_{(Y,\eta)}(e, 0)) \rangle) (m).$$

Next, we compute $\langle \mathcal{L}_{\tilde{X}} \tilde{\eta}, T_m e Z(m) \rangle$. Using Lemma C.4 and the identity $\phi_t^{\tilde{X}}(e_m) = \psi_{-t}^X(e)(\phi_t^X(m))$, we find

$$\begin{aligned} \langle \mathcal{L}_{\tilde{X}} \tilde{\eta}(e_m), T_m e Z(m) \rangle &= \frac{d}{dt} \Big|_{t=0} \langle \tilde{\eta}_{\phi_t^{\tilde{X}}(e_m)} \circ T_{e_m} \phi_t^{\tilde{X}}, T_m e Z(m) \rangle \\ &= \frac{d}{dt} \Big|_{t=0} \langle \tilde{\eta}_{\psi_{-t}^X(e)(\phi_t^X(m))}, T_{\phi_t^X(m)} \psi_{-t}^X(e)((\phi_{-t}^X)^*(Z)(\phi_t^X(m))) \rangle \\ &= \frac{d}{dt} \Big|_{t=0} \langle ((\phi_{-t}^X)^*(Z)) \langle \eta, \psi_{-t}^X(e) \rangle (\phi_t^X(m)) \\ &\quad - \frac{d}{dt} \Big|_{t=0} \langle \text{pr}_{T^*M} \Delta_{(Y,\eta)}(\psi_{-t}^X(e), 0), (\phi_{-t}^X)^*(Z) \rangle (\phi_t^X(m)) \\ &= (-[X, Z] \langle \eta, e \rangle + XZ \langle \eta, e \rangle - Z \langle \eta, \text{pr}_E \Delta_{(X,\xi)}(e, 0) \rangle \\ &\quad - X \langle \text{pr}_{T^*M} \Delta_{(Y,\eta)}(e, 0), Z \rangle + \langle \text{pr}_{T^*M} \Delta_{(Y,\eta)}(e, 0), [X, Z] \rangle \\ &\quad + \langle \text{pr}_{T^*M} \Delta_{(Y,\eta)}(\text{pr}_E \Delta_{(X,\xi)}(e, 0), 0), Z \rangle) (m). \end{aligned}$$

We have also

$$\begin{aligned} \langle \mathbf{d}_{e_m} \langle \tilde{\xi}, \tilde{Y} \rangle, T_m e Z(m) \rangle &= Z \left(\langle \tilde{\xi}, \tilde{Y} \rangle \circ e \right) \\ &= Z \left(Y \langle \xi, e \rangle - \langle \text{pr}_{T^*M} \Delta_{(X,\xi)}(e, 0), Y \rangle - \langle \xi, \text{pr}_E \Delta_{(Y,\eta)}(e, 0) \rangle \right), \end{aligned}$$

which leads to

$$\begin{aligned}
& \langle \mathcal{L}_{\tilde{X}} \tilde{\eta} - \mathbf{i}_{\tilde{Y}} \mathbf{d}\tilde{\xi}, T_m e Z(m) \rangle \\
&= \langle \mathcal{L}_{\tilde{X}} \tilde{\eta} - \mathcal{L}_{\tilde{Y}} \tilde{\xi} + \mathbf{d}\langle \tilde{\xi}, \tilde{Y} \rangle, T_m e Z(m) \rangle \\
&= \langle ZX\langle \eta, e \rangle - Z\langle \eta, \text{pr}_E \Delta_{(X,\xi)}(e, 0) \rangle \\
&\quad - X\langle \text{pr}_{T^*M} \Delta_{(Y,\eta)}(e, 0), Z \rangle + \langle \text{pr}_{T^*M} \Delta_{(Y,\eta)}(e, 0), [X, Z] \rangle \\
&\quad + \langle \text{pr}_{T^*M} \Delta_{(Y,\eta)}(\text{pr}_E \Delta_{(X,\xi)}(e, 0), 0), Z \rangle \\
&\quad - ZY\langle \xi, e \rangle + Z\langle \xi, \text{pr}_E \Delta_{(Y,\eta)}(e, 0) \rangle \\
&\quad + Y\langle \text{pr}_{T^*M} \Delta_{(X,\xi)}(e, 0), Z \rangle - \langle \text{pr}_{T^*M} \Delta_{(X,\xi)}(e, 0), [Y, Z] \rangle \\
&\quad - \langle \text{pr}_{T^*M} \Delta_{(X,\xi)}(\text{pr}_E \Delta_{(Y,\eta)}(e, 0), 0), Z \rangle \\
&\quad + ZY\langle \xi, e \rangle - Z\langle \text{pr}_{T^*M} \Delta_{(X,\xi)}(e, 0), Y \rangle - Z\langle \xi, \text{pr}_E \Delta_{(Y,\eta)}(e, 0) \rangle \rangle (m) \\
&= \langle Z\langle \llbracket (X, \xi), (Y, \eta) \rrbracket_\Delta, (e, 0) \rangle \\
&\quad - X\langle \Delta_{(Y,\eta)}(e, 0), (Z, 0) \rangle + \langle \text{pr}_{T^*M} \Delta_{(Y,\eta)}(e, 0), [X, Z] \rangle \\
&\quad + \langle \Delta_{(Y,\eta)}(\text{pr}_E \Delta_{(X,\xi)}(e, 0), 0), (Z, 0) \rangle \\
&\quad + Y\langle \Delta_{(X,\xi)}(e, 0), (Z, 0) \rangle - \langle \text{pr}_{T^*M} \Delta_{(X,\xi)}(e, 0), [Y, Z] \rangle \\
&\quad - \langle \Delta_{(X,\xi)}(\text{pr}_E \Delta_{(Y,\eta)}(e, 0), 0), (Z, 0) \rangle \rangle (m) \\
&= \langle Z\langle \llbracket (X, \xi), (Y, \eta) \rrbracket_\Delta, (e, 0) \rangle \\
&\quad - \langle \Delta_{(X,\xi)} \Delta_{(Y,\eta)}(e, 0), (Z, 0) \rangle - \langle \Delta_{(Y,\eta)}(e, 0), \llbracket (X, \xi), (Z, 0) \rrbracket_\Delta \rangle \\
&\quad + \langle \text{pr}_{T^*M} \Delta_{(Y,\eta)}(e, 0), [X, Z] \rangle - \langle (\text{pr}_E \Delta_{(X,\xi)}(e, 0), 0), \llbracket (Y, \eta), (Z, 0) \rrbracket_\Delta \rangle \\
&\quad + \langle \Delta_{(Y,\eta)} \Delta_{(X,\xi)}(e, 0), (Z, 0) \rangle + \langle \Delta_{(X,\xi)}(e, 0), \llbracket (Y, \eta), (Z, 0) \rrbracket_\Delta \rangle \\
&\quad - \langle \text{pr}_{T^*M} \Delta_{(X,\xi)}(e, 0), [Y, Z] \rangle \\
&\quad + \langle (\text{pr}_E \Delta_{(Y,\eta)}(e, 0), 0), \llbracket (X, \xi), (Z, 0) \rrbracket_\Delta \rangle \rangle (m) \\
&= \langle Z\langle \llbracket (X, \xi), (Y, \eta) \rrbracket_\Delta, (e, 0) \rangle - \langle \Delta_{(X,\xi)} \Delta_{(Y,\eta)}(e, 0) - \Delta_{(Y,\eta)} \Delta_{(X,\xi)}(e, 0), (Z, 0) \rangle \rangle (m).
\end{aligned}$$

For the last equality, we have used Lemma C.2 in the case $\chi = 0$. Now we can conclude:

$$\begin{aligned}
\langle (g, \omega), (Z, \chi) \rangle (m) &= \langle \chi, \text{pr}_E (\Delta_{(Y,\eta)} \Delta_{(X,\xi)}(e, 0) - \Delta_{(X,\xi)} \Delta_{(Y,\eta)}(e, 0)) \rangle (m) \\
&\quad - \langle \Delta_{(X,\xi)} \Delta_{(Y,\eta)}(e, 0) - \Delta_{(Y,\eta)} \Delta_{(X,\xi)}(e, 0), (Z, 0) \rangle (m) \\
&= - \langle \Delta_{(X,\xi)} \Delta_{(Y,\eta)}(e, 0) - \Delta_{(Y,\eta)} \Delta_{(X,\xi)}(e, 0), (Z, \chi) \rangle (m).
\end{aligned}$$

This shows that

$$(g, \omega)(m) = -R_\Delta((X, \xi), (Y, \eta))(e, 0)(m) - \Delta_{\llbracket (X, \xi), (Y, \eta) \rrbracket_\Delta}(e, 0)(m).$$

□

APPENDIX D. THE LIE ALGEBROID STRUCTURE ON $TA \oplus T^*A \rightarrow TM \oplus A^*$

Let $(q_A : A \rightarrow M, \rho, [\cdot, \cdot])$ be a Lie algebroid. We describe here the Lie algebroid structures on $TA \rightarrow TM$, $T^*A \rightarrow A^*$ and $TA \oplus T^*A \rightarrow TM \oplus A^*$.

For simplicity, we will write $q := q_A : A \rightarrow M$ and $q_* := q_{A^*} : A^* \rightarrow M$ for the vector bundle maps.

The Lie algebroid $TA \rightarrow TM$. For $a \in \Gamma(A)$, we have two particular types of sections of $TA \rightarrow TM$: the *linear* sections $Ta : TM \rightarrow TA$, which are vector bundle morphisms over $a : M \rightarrow A$, and the *core* sections $a^\dagger : TM \rightarrow TA$, $a^\dagger(x_m) = T_m 0^A x_m +_{p_A} \frac{d}{dt} \Big|_{t=0} t \cdot a(m)$.

The Lie algebroid structure of $TA \rightarrow TM$ is given by

$$\begin{aligned} [Ta, Tb] &= T[a, b] \\ [Ta, b^\dagger] &= [a, b]^\dagger \\ [a^\dagger, b^\dagger] &= 0 \\ \rho_{TA}(Ta) &= [\widehat{\rho(a)}, \cdot] \in \mathfrak{X}(TM) \\ \rho_{TA}(a^\dagger) &= (\rho(a))^\dagger \in \mathfrak{X}(TM). \end{aligned}$$

That is, we have for all $a \in \Gamma(A)$, $\varphi \in C^\infty(M)$ and $\theta \in \Omega^1(M)$:

$$\begin{aligned} \rho_{TA}(Ta)(p_M^* \varphi) &= p_M^*(\rho(a)(\varphi)) \\ \rho_{TA}(Ta)(\ell_\theta) &= \ell_{\mathcal{L}_{\rho(a)} \theta} \\ \rho_{TA}(a^\dagger)(p_M^* \varphi) &= 0 \\ \rho_{TA}(a^\dagger)(\ell_\theta) &= p_M^* \langle \theta, \rho(a) \rangle \end{aligned}$$

(see for instance [22]).

The Lie algebroid $T^*A \rightarrow A^*$. There is an isomorphism of double vector bundles

$$\begin{array}{ccc} T^*A^* & \xrightarrow{r_{A^*}} & A \\ c_{A^*} \downarrow & & \downarrow \\ A^* & \longrightarrow & M \end{array} \xrightarrow{R} \begin{array}{ccc} T^*A & \xrightarrow{c_A} & A \\ r_A \downarrow & & \downarrow \\ A^* & \longrightarrow & M \end{array}$$

over the identity on the sides, and $-\text{id}_{T^*M}$ on the core T^*M . The map R is given as follows: for $\omega \in \Omega^1(M)$, we have

$$R(q_*^* \omega(\xi_m)) = \mathbf{d}_{0_m^A} \ell_\xi - q^* \omega(0_m^A)$$

and for $\xi \in \Gamma(A^*)$ and $a \in \Gamma(A)$, we have

$$R(\mathbf{d}_{\xi(m)} \ell_a) = \mathbf{d}_{a(m)} (\ell_\xi - q^* \langle \xi, a \rangle)$$

for all $m \in M$. In the following, we will write $\omega^\dagger \in \Gamma_{A^*}(T^*A)$ for the section $\xi_m \mapsto R(q_*^* \omega(\xi_m))$, and $a^\dagger \in \Gamma_{A^*}(T^*A)$ for the section $\xi_m \mapsto R(\mathbf{d}_{\xi(m)} \ell_a)$.

Recall that since A is a Lie algebroid, its dual A^* is endowed with a linear Poisson structure given by

$$\begin{aligned} \{\ell_a, \ell_b\} &= \ell_{[a, b]} \\ \{\ell_a, q_*^* f\} &= q_*^*(\rho(a)(f)) \\ \{q_*^* \varphi, q_*^* \psi\} &= 0 \end{aligned}$$

for all $a, b \in \Gamma(A)$ and $\varphi, \psi \in C^\infty(M)$. Hence, there is a Lie algebroid structure on $T^*A^* \rightarrow A^*$ associated to this Poisson structure, and the Lie algebroid structure on $T^*A \rightarrow A^*$ is exactly such that the isomorphism

$$R : T^*A^* \rightarrow T^*A$$

is an isomorphism of Lie algebroids [23, 24].

Therefore, we give first the Lie brackets and images under the anchor map $\rho_{T^*A^*}$ of the sections \mathbf{d}_a and $q_*^* \omega \in \Omega^1(A^*) = \Gamma_{A^*}(T^*A^*)$, for $\omega \in \Omega^1(M)$ and $a \in \Gamma(A)$.

By the definition of the Lie algebroid structure $T^*A^* \rightarrow A^*$ associated to the linear Poisson structure on A^* , one gets easily that the Lie algebroid structure on $T^*A^* \rightarrow A^*$ is

given by the following identities:

$$\begin{aligned}
[\mathbf{d}\ell_a, \mathbf{d}\ell_b] &= \mathbf{d}\ell_{[a,b]} \\
[\mathbf{d}\ell_a, q_*^*\theta] &= q_*^*(\mathcal{L}_{\rho(a)}\theta) \\
[q_*^*\theta, q_*^*\omega] &= 0 \\
\rho_{T^*A^*}(\mathbf{d}\ell_a) &= \widehat{\mathcal{L}}_a \in \mathfrak{X}(A^*) \\
\rho_{T^*A^*}(q_*^*\theta) &= (\rho^*\theta)^\dagger \in \mathfrak{X}(A^*)
\end{aligned}$$

for $a, b \in \Gamma(A)$ and $\theta, \omega \in \Omega^1(M)$.

As a consequence, we find that the Lie algebroid structure on $T^*A \rightarrow A^*$ is given by

$$\begin{aligned}
[a^l, b^l] &= [a, b]^l \\
[a^l, \theta^\dagger] &= q^*(\mathcal{L}_{\rho(a)}\theta) \\
[\theta^\dagger, \omega^\dagger] &= 0 \\
\rho_{T^*A}(a^l) &= \widehat{\mathcal{L}}_a \in \mathfrak{X}(A^*) \\
\rho_{T^*A}(\theta^\dagger) &= (\rho^*\theta)^\dagger \in \mathfrak{X}(A^*)
\end{aligned}$$

for $a, b \in \Gamma(A)$ and $\theta, \omega \in \Omega^1(M)$.

The fibered product $TA \times_A T^*A \rightarrow TM \times_M A^*$. The Lie algebroid $TA \oplus T^*A \rightarrow TM \oplus A^*$ is defined as the pullback to the diagonals $\Delta_A \rightarrow \Delta_M$ of the Lie algebroid $TA \times T^*A \rightarrow TM \times A^*$. We have the special sections

$$\tilde{a} := (Ta, a^l) : TM \oplus A^* \rightarrow TA \oplus T^*A$$

for $a \in \Gamma(A)$ and

$$(b, \theta)^\dagger := (b^\dagger, \theta^\dagger) : TM \oplus A^* \rightarrow TA \oplus T^*A$$

for $(b, \theta) \in \Gamma(A \oplus T^*M)$. The set of sections of $TA \oplus T^*A \rightarrow TM \oplus A^*$ is spanned as a $C^\infty(TM \oplus A^*)$ -module by these two types of sections. Given a vector bundle morphism $\Phi \in \Gamma(\text{Hom}(TM \oplus A^*, A \oplus T^*M))$, we can define the section

$$\Phi^\dagger : TM \oplus A^* \rightarrow TA \oplus T^*A,$$

by

$$\Phi^\dagger(x_m, \xi_m) = (\Phi(x_m, \xi_m))^\dagger(x_m, \xi_m)$$

for all $(x_m, \xi_m) \in TM \oplus A^*$.

We will write $\pi : TM \oplus A^* \rightarrow M$ for the projection and $\Theta : TA \oplus T^*A \rightarrow T(TM \oplus A^*)$ for the anchor of $TA \oplus T^*A \rightarrow TM \oplus A^*$.

Proposition D.1. *The Lie algebroid $(TA \oplus T^*A, \Theta, [\cdot, \cdot])$ is described by the following identities*

$$\begin{aligned}
[\tilde{a}, \tilde{b}] &= [\widetilde{a, b}] \\
[\tilde{a}, \sigma^\dagger] &= (\mathcal{L}_a \sigma)^\dagger \\
[\sigma^\dagger, \tau^\dagger] &= 0 \\
[\tilde{a}, \Phi^\dagger] &= (\mathcal{L}_a \Phi)^\dagger \\
[\sigma^\dagger, \Phi^\dagger] &= \Phi((\rho, \rho^*)\sigma)^\dagger \\
[\Phi^\dagger, \Psi^\dagger] &= (\Psi \circ (\rho, \rho^*) \circ \Phi - \Phi \circ (\rho, \rho^*) \circ \Psi)^\dagger \\
\Theta(\tilde{a}) &= \widehat{\mathcal{L}_a} \\
\Theta(\sigma^\dagger) &= ((\rho, \rho^*)\sigma)^\dagger \\
\Theta(\Phi^\dagger) &= ((\rho, \rho^*) \circ \Phi)^\dagger
\end{aligned}$$

for $a, b \in \Gamma(A)$, $\sigma, \tau \in \Gamma(A \oplus T^*M)$ and $\Phi, \Psi \in \Gamma(\text{Hom}(TM \oplus A^*, A \oplus T^*M))$.

Proof. We start by computing the anchor. Note first that for all $\varphi \in C^\infty(M)$ and $\tau = (c, \omega) \in \Gamma(A \oplus T^*M)$, we have $\pi^*\varphi = \text{pr}_{TM}^* p_M^* \varphi = \text{pr}_{A^*}^* q_*^* \varphi$ and $\ell_\tau = \text{pr}_{TM}^* \ell_\omega + \text{pr}_{A^*}^* \ell_c$. Thus, we get:

$$\begin{aligned}
\Theta(\tilde{a})(\ell_\tau) &= (\rho_{TA} \circ Ta, \rho_{T^*A} \circ R(\mathbf{d}\ell_a))(\text{pr}_{TM}^* \ell_\omega + \text{pr}_{A^*}^* \ell_c) \\
&= \text{pr}_{TM}^* (\rho_{TA} \circ Ta)(\ell_\omega) + \text{pr}_{A^*}^* (\rho_{T^*A} \circ R(\mathbf{d}\ell_a))(\ell_c) \\
&= \text{pr}_{TM}^* \ell_{\mathcal{L}_{\rho(a)}\omega} + \text{pr}_{A^*}^* \ell_{[a, c]} = \ell_{\mathcal{L}_a \tau}, \\
\Theta(\tilde{a})(\pi^* \varphi) &= (\rho_{TA} \circ Ta, \rho_{T^*A} \circ R(\mathbf{d}\ell_a))(\text{pr}_{TM}^* p_M^* \varphi) \\
&= \text{pr}_{TM}^* p_M^* (\rho(a)(\varphi)) = \pi^* (\rho(a)(\varphi)), \\
\Theta(\sigma^\dagger)(\ell_\tau) &= (\rho_{TA} \circ b^\dagger, \rho_{T^*A} \circ q_*^* \theta)(\text{pr}_{TM}^* \ell_\omega + \text{pr}_{A^*}^* \ell_c) \\
&= \text{pr}_{TM}^* (\rho_{TA} \circ b^\dagger)(\ell_\omega) + \text{pr}_{A^*}^* (\rho_{T^*A} \circ q_*^* \theta)(\ell_c) \\
&= \text{pr}_{TM}^* p_M^* \langle \omega, \rho(b) \rangle + \text{pr}_{A^*}^* q_*^* \langle \theta, \rho(c) \rangle = \pi^* \langle (\rho, \rho^*)\sigma, \tau \rangle, \\
\Theta(\sigma^\dagger)(\pi^* \varphi) &= (\rho_{TA} \circ b^\dagger, \rho_{T^*A} \circ q_*^* \theta)(\text{pr}_{TM}^* p_M^* \varphi) = 0.
\end{aligned}$$

For the last equality, note that a section $\Phi \in \Gamma(\text{Hom}(TM \oplus A^*, A \oplus T^*M))$ can be written as a sum $\Phi = \sum_{i=1}^n \varphi^i \cdot \ell_{\sigma_i} \cdot \tau_i$ with $\varphi_i \in C^\infty(M)$ and $\sigma_i, \tau_i \in \Gamma(A \oplus T^*M)$, for $i = 1, \dots, n$. The corresponding section $\Phi^\dagger \in \Gamma_{TM \oplus A^*}(TA \oplus T^*A)$ is then given by

$$\Phi^\dagger = \sum_{i=1}^n \pi^* \varphi^i \cdot \ell_{\sigma_i} \cdot \tau_i^\dagger$$

and we get

$$\Theta(\Phi^\dagger) = \sum_{i=1}^n \pi^* \varphi^i \cdot \ell_{\sigma_i} \cdot \Theta(\tau_i^\dagger) = \sum_{i=1}^n \pi^* \varphi^i \cdot \ell_{\sigma_i} \cdot ((\rho, \rho^*)\tau_i)^\dagger = ((\rho, \rho^*) \circ \Phi)^\dagger.$$

Next we compute the Lie algebroid brackets. For $a, b \in \Gamma(A)$ and $\theta, \omega \in \Omega^1(M)$, we have $[(Ta, R(\mathbf{d}\ell_a)), (Tb, R(\mathbf{d}\ell_b))] = (T[a, b], R(\mathbf{d}\ell_{[a, b]}))$ by the considerations in the previous sections. In the same manner, we show the next two identities: $[\tilde{a}, (b, \theta)^\dagger] = ([a, b], \mathcal{L}_{\rho(a)}\theta)^\dagger$ and $[(a, \theta)^\dagger, (b, \omega)^\dagger] = (0, 0)$. This yields for the last three brackets, assuming without loss of generality that $\Phi = \varphi \cdot \ell_{\sigma_1} \cdot \tau_1$ and $\Psi = \psi \cdot \ell_{\sigma_2} \cdot \tau_2$ with $\varphi, \psi \in C^\infty(M)$ and $\sigma_1, \sigma_2, \tau_1, \tau_2 \in$

$\Gamma(A \oplus T^*M)$:

$$\begin{aligned} [\tilde{a}, \Phi^\dagger] &= [\tilde{a}, \pi^* \varphi \cdot \ell_{\sigma_1} \cdot \tau_1^\dagger] \\ &= \pi^*(\rho(a)(\varphi)) \cdot \ell_{\sigma_1} \cdot \tau_1^\dagger + \pi^* \varphi \cdot \ell_{\mathcal{L}_a \sigma_1} \cdot \tau_1^\dagger + \pi^* \varphi \cdot \ell_{\sigma_1} \cdot (\mathcal{L}_a \tau_1)^\dagger = (\mathcal{L}_a \Phi)^\dagger, \end{aligned}$$

since

$$\begin{aligned} (\mathcal{L}_a \Phi)(v) &= \mathcal{L}_a(\Phi(v)) - \Phi(\mathcal{L}_a(v)) = \mathcal{L}_a(\varphi \cdot \langle \sigma_1, v \rangle \cdot \tau_1) - \varphi \cdot \langle \sigma_1, \mathcal{L}_a(v) \rangle \cdot \tau_1 \\ &= \rho(a)(\varphi) \cdot \langle \sigma_1, v \rangle \cdot \tau_1 + \varphi \cdot \langle \mathcal{L}_a \sigma_1, v \rangle \cdot \tau_1 + \varphi \cdot \langle \sigma_1, v \rangle \cdot \mathcal{L}_a \tau_1 \end{aligned}$$

for $v \in \Gamma(TM \oplus A^*)$. We finish the proof with

$$\begin{aligned} [\sigma_2^\dagger, \Phi^\dagger] &= [\sigma_2^\dagger, \pi^* \varphi \cdot \ell_{\sigma_1} \cdot \tau_1^\dagger] \\ &= \pi^* \varphi \cdot \pi^* \langle (\rho, \rho^*) \sigma_2, \sigma_1 \rangle \cdot \tau_1^\dagger = \Phi((\rho, \rho^*) \sigma_2)^\dagger \\ [\Phi^\dagger, \Psi^\dagger] &= [\pi^* \varphi \cdot \ell_{\sigma_1} \cdot \tau_1^\dagger, \pi^* \psi \cdot \ell_{\sigma_2} \cdot \tau_2^\dagger] \\ &= \pi^* \varphi \cdot \ell_{\sigma_1} \cdot \Psi((\rho, \rho^*) \tau_1)^\dagger - \pi^* \psi \cdot \ell_{\sigma_2} \cdot \Phi((\rho, \rho^*) \tau_2)^\dagger \\ &= (\Psi \circ (\rho, \rho^*) \circ \Phi - \Phi \circ (\rho, \rho^*) \circ \Psi)^\dagger. \end{aligned}$$

□

APPENDIX E. PROOFS OF THE MAIN THEOREMS IN SECTIONS 6.1 AND 6.2

Proof of Theorem 6.3. We start by checking that $\llbracket \cdot, \cdot \rrbracket$ is well-defined. Choose $k = (a, \theta) \in \Gamma(K)$, $\sigma = (b, \omega) \in \Gamma(A \oplus T^*M)$ and $u = (X, \xi) \in \Gamma(U)$. We have then

$$\begin{aligned} \llbracket u \oplus \sigma, (\rho, \rho^*)(-k) \oplus k \rrbracket &= (-\llbracket u, (\rho, \rho^*)k \rrbracket_\Delta - \nabla_b^{\text{bas}}(\rho, \rho^*)k - \nabla_a^{\text{bas}}u) \\ &\quad \oplus ([\sigma, k]_D + \Delta_u k + \Delta_{(\rho, \rho^*)k} \sigma - (0, \mathbf{d}\langle \sigma, (\rho, \rho^*)k \rangle)). \end{aligned}$$

Using Lemma 5.15, we see immediately that the second term of this sum equals

$$\nabla_b^{\text{bas}}k + \Delta_u k.$$

Since (K, U, Δ) is an \mathcal{LA} -triple, we know that $\nabla_b^{\text{bas}}k \in \Gamma(K)$, and $\Delta_u k \in \Gamma(K)$ follows from the fact that (K, U, Δ) is Dirac.

By Proposition 5.2, we have

$$\nabla_a^{\text{bas}}(\rho, \rho^*)k = (\rho, \rho^*)\nabla_a^{\text{bas}}k$$

and by Lemma 5.17:

$$(\rho, \rho^*)\Delta_u k = \llbracket u, (\rho, \rho^*)k \rrbracket_\Delta + \nabla_b^{\text{bas}}u.$$

Thus, we have shown that

$$\llbracket u \oplus \sigma, (\rho, \rho^*)(-k) \oplus k \rrbracket = (-\nabla_a^{\text{bas}}k - \Delta_u k) \oplus (\nabla_a^{\text{bas}}k + \Delta_u k),$$

which is 0 in C . The equality

$$\llbracket (\rho, \rho^*)(-k) \oplus k, u \oplus \sigma \rrbracket = 0$$

follows with (4) below and

$$\langle \langle u \oplus \sigma, (\rho, \rho^*)(-k) \oplus k \rangle \rangle_C = \langle u, k \rangle = 0$$

since $U = K^\circ$.

In the following, we will write $u = (X, \xi)$, $u_i = (X_i, \xi_i)$ and $\sigma = (a, \theta)$, $\sigma_i = (a_i, \theta_i)$, $i = 1, 2, 3$. We check (1) – (3) in the definition of a Courant algebroid, in reverse order

(3) We have

$$\begin{aligned}
& \llbracket u_1 \oplus \sigma_1, u_2 \oplus \sigma_2 \rrbracket_C + \llbracket u_2 \oplus \sigma_2, u_1 \oplus \sigma_1 \rrbracket_C \\
&= (\llbracket u_1, u_2 \rrbracket_\Delta + \llbracket u_2, u_1 \rrbracket_\Delta) \oplus ([\sigma_1, \sigma_2]_D + [\sigma_2, \sigma_1]_D + (0, \mathbf{d}\langle \sigma_1, u_2 \rangle + \mathbf{d}\langle \sigma_2, u_1 \rangle)) \\
&= 0 \oplus (0, \mathbf{d}(\langle \sigma_1, \sigma_2 \rangle_D + \langle \sigma_1, u_2 \rangle + \langle \sigma_2, u_1 \rangle)) \\
&= \mathcal{D}\langle u_1 \oplus \sigma_1, u_2 \oplus \sigma_2 \rangle_C.
\end{aligned}$$

(2) We then check the equality

$$\begin{aligned}
& c(u_1 \oplus \sigma_1) \langle u_2 \oplus \sigma_2, u_3 \oplus \sigma_3 \rangle_C \\
&= \langle \llbracket u_1 \oplus \sigma_1, u_2 \oplus \sigma_2 \rrbracket_C, u_3 \oplus \sigma_3 \rangle_C + \langle u_2 \oplus \sigma_2, \llbracket u_1 \oplus \sigma_1, u_3 \oplus \sigma_3 \rrbracket_C \rangle_C.
\end{aligned}$$

We have

$$\begin{aligned}
& \langle \llbracket u_1 \oplus \sigma_1, u_2 \oplus \sigma_2 \rrbracket, u_3 \oplus \sigma_3 \rangle_C \\
&= \langle \langle \llbracket u_1, u_2 \rrbracket_\Delta + \nabla_{\sigma_1}^{\text{bas}} u_2 - \nabla_{\sigma_2}^{\text{bas}} u_1 \rangle \oplus ([\sigma_1, \sigma_2]_D + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, u_2 \rangle)), u_3 \oplus \sigma_3 \rangle_C \\
&\stackrel{(5.15)}{=} \langle \langle \llbracket u_1, u_2 \rrbracket_\Delta + \nabla_{\sigma_1}^{\text{bas}} u_2 - \nabla_{\sigma_2}^{\text{bas}} u_1 \rangle \oplus (\nabla_{\sigma_1}^{\text{bas}} \sigma_2 - \Delta_{(\rho, \rho^*) \sigma_2} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, (\rho, \rho^*) \sigma_2 \rangle) \\
&\quad + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, u_2 \rangle)), u_3 \oplus \sigma_3 \rangle_C \\
&= \langle \llbracket u_1, u_2 \rrbracket_\Delta + \nabla_{\sigma_1}^{\text{bas}} u_2 - \nabla_{\sigma_2}^{\text{bas}} u_1, \sigma_3 \rangle + X_3 \langle \sigma_1, u_2 \rangle + X_3 \langle \sigma_1, (\rho, \rho^*) \sigma_2 \rangle \\
&\quad + \langle \nabla_{\sigma_1}^{\text{bas}} \sigma_2 - \Delta_{(\rho, \rho^*) \sigma_2} \sigma_1 + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1, u_3 \rangle + \rho(a_3) \langle \sigma_1, u_2 \rangle \\
&\quad + \rho(a_3) \langle \sigma_1, (\rho, \rho^*) \sigma_2 \rangle + \langle \nabla_{\sigma_1}^{\text{bas}} \sigma_2 - \Delta_{(\rho, \rho^*) \sigma_2} \sigma_1 + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1, (\rho, \rho^*) \sigma_3 \rangle.
\end{aligned}$$

Hence, we can compute

$$\begin{aligned}
& \langle \llbracket u_1 \oplus \sigma_1, u_2 \oplus \sigma_2 \rrbracket, u_3 \oplus \sigma_3 \rangle_C + \langle \llbracket u_1 \oplus \sigma_1, u_3 \oplus \sigma_3 \rrbracket, u_2 \oplus \sigma_2 \rangle_C \\
&= X_1 \langle u_2, \sigma_3 \rangle - \langle u_2, \Delta_{u_1} \sigma_3 \rangle + \langle \nabla_{\sigma_1}^{\text{bas}} u_2 - \nabla_{\sigma_2}^{\text{bas}} u_1, \sigma_3 \rangle + X_3 \langle \sigma_1, u_2 \rangle + X_3 \langle \sigma_1, (\rho, \rho^*) \sigma_2 \rangle \\
&\quad + \langle \nabla_{\sigma_1}^{\text{bas}} \sigma_2 - \Delta_{(\rho, \rho^*) \sigma_2} \sigma_1 + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1, u_3 \rangle + \rho(a_3) \langle \sigma_1, u_2 \rangle \\
&\quad + \rho(a_3) \langle \sigma_1, (\rho, \rho^*) \sigma_2 \rangle + \langle \nabla_{\sigma_1}^{\text{bas}} \sigma_2 - \Delta_{(\rho, \rho^*) \sigma_2} \sigma_1 - \Delta_{u_2} \sigma_1, (\rho, \rho^*) \sigma_3 \rangle \\
&\quad + X_1 \langle \sigma_2, (\rho, \rho^*) \sigma_3 \rangle - \langle \sigma_2, \llbracket u_1, (\rho, \rho^*) \sigma_3 \rrbracket_\Delta \rangle \\
&\quad + X_1 \langle u_3, \sigma_2 \rangle - \langle u_3, \Delta_{u_1} \sigma_2 \rangle + \langle \nabla_{\sigma_1}^{\text{bas}} u_3 - \nabla_{\sigma_3}^{\text{bas}} u_1, \sigma_2 \rangle + X_2 \langle \sigma_1, u_3 \rangle + X_2 \langle \sigma_1, (\rho, \rho^*) \sigma_3 \rangle \\
&\quad + \langle \nabla_{\sigma_1}^{\text{bas}} \sigma_3 - \Delta_{(\rho, \rho^*) \sigma_3} \sigma_1 + \Delta_{u_1} \sigma_3 - \Delta_{u_3} \sigma_1, u_2 \rangle + \rho(a_2) \langle \sigma_1, u_3 \rangle \\
&\quad + \rho(a_2) \langle \sigma_1, (\rho, \rho^*) \sigma_3 \rangle + \langle \nabla_{\sigma_1}^{\text{bas}} \sigma_3 - \Delta_{(\rho, \rho^*) \sigma_3} \sigma_1 + \Delta_{u_1} \sigma_3 - \Delta_{u_3} \sigma_1, (\rho, \rho^*) \sigma_2 \rangle.
\end{aligned}$$

Using the duality of the Dorfman connections and the corresponding dull bracket, and the identities in Proposition 5.2, we get

$$\begin{aligned}
& \langle \llbracket u_1 \oplus \sigma_1, u_2 \oplus \sigma_2 \rrbracket, u_3 \oplus \sigma_3 \rangle_C + \langle \llbracket u_1 \oplus \sigma_1, u_3 \oplus \sigma_3 \rrbracket, u_2 \oplus \sigma_2 \rangle_C \\
&= X_1 \langle \llbracket u_2 \oplus \sigma_2, u_3 \oplus \sigma_3 \rrbracket \rangle_C + \langle \sigma_1, \llbracket u_2, u_3 \rrbracket_\Delta \rangle + \langle \sigma_1, \llbracket u_3, u_2 \rrbracket_\Delta \rangle \\
&\quad + \langle \sigma_1, \llbracket (\rho, \rho^*) \sigma_2, u_3 \rrbracket_\Delta \rangle + \langle \sigma_1, \llbracket u_3, (\rho, \rho^*) \sigma_2 \rrbracket_\Delta \rangle \\
&\quad + \langle \sigma_1, \llbracket (\rho, \rho^*) \sigma_3, u_2 \rrbracket_\Delta \rangle + \langle \sigma_1, \llbracket u_2, (\rho, \rho^*) \sigma_3 \rrbracket_\Delta \rangle \\
&\quad + \langle \sigma_1, \llbracket (\rho, \rho^*) \sigma_3, (\rho, \rho^*) \sigma_2 \rrbracket_\Delta \rangle \\
&\quad + \rho(a_1) \langle u_2, \sigma_3 \rangle - \langle \text{Skew}_\Delta(u_2, (\rho, \rho^*) \sigma_3), a_1 \rangle \\
&\quad + \rho(a_1) \langle u_3, \sigma_2 \rangle - \langle \text{Skew}_\Delta(u_3, (\rho, \rho^*) \sigma_2), a_1 \rangle \\
&\quad + \rho(a_1) \langle \sigma_2, (\rho, \rho^*) \sigma_3 \rangle - \langle \text{Skew}_\Delta((\rho, \rho^*) \sigma_3, (\rho, \rho^*) \sigma_2), a_1 \rangle \\
&\quad - \langle \nabla_{\sigma_2}^{\text{bas}} u_1, \sigma_3 \rangle - \langle \Delta_{(\rho, \rho^*) \sigma_2} \sigma_1, (\rho, \rho^*) \sigma_3 \rangle - \langle \sigma_2, \llbracket u_1, (\rho, \rho^*) \sigma_3 \rrbracket_\Delta \rangle \\
&\quad - \langle \nabla_{\sigma_3}^{\text{bas}} u_1, \sigma_2 \rangle + \rho(a_2) \langle \sigma_1, (\rho, \rho^*) \sigma_3 \rangle + \langle \Delta_{u_1} \sigma_3, (\rho, \rho^*) \sigma_2 \rangle \\
&= c(u_1 \oplus \sigma_1) \langle \llbracket u_2 \oplus \sigma_2, u_3 \oplus \sigma_3 \rrbracket \rangle_C - \langle \sigma_1, \llbracket (\rho, \rho^*) \sigma_2, (\rho, \rho^*) \sigma_3 \rrbracket_\Delta \rangle \\
&\quad - \langle \nabla_{\sigma_2}^{\text{bas}} u_1, \sigma_3 \rangle - \langle \Delta_{(\rho, \rho^*) \sigma_2} \sigma_1, (\rho, \rho^*) \sigma_3 \rangle + \rho(a_2) \langle \sigma_1, (\rho, \rho^*) \sigma_3 \rangle \\
&\quad + \langle \sigma_2, (\rho, \rho^*) \Delta_{u_1} \sigma_3 - \llbracket u_1, (\rho, \rho^*) \sigma_3 \rrbracket_\Delta - \nabla_{\sigma_3}^{\text{bas}} u_1 \rangle \\
&= c(u_1 \oplus \sigma_1) \langle \llbracket u_2 \oplus \sigma_2, u_3 \oplus \sigma_3 \rrbracket \rangle_C.
\end{aligned}$$

For the second equality, we have used the definition of Skew_Δ and the fact that $\llbracket \cdot, \cdot \rrbracket_\Delta$ is skew-symmetric on sections of U . For the last equality, we have used Lemma 5.16 and the duality equation for Δ and $\llbracket \cdot, \cdot \rrbracket_\Delta$.

(1) For the Jacobi identity, we will check that

$$\begin{aligned}
& \text{Jac}_{\llbracket \cdot, \cdot \rrbracket_\Delta}(u_1 \oplus \sigma_1, u_2 \oplus \sigma_2, u_3 \oplus \sigma_3) \\
&:= \llbracket \llbracket u_1 \oplus \sigma_1, u_2 \oplus \sigma_2 \rrbracket, u_3 \oplus \sigma_3 \rrbracket + \llbracket u_2 \oplus \sigma_2, \llbracket u_1 \oplus \sigma_1, u_3 \oplus \sigma_3 \rrbracket \rrbracket \\
&\quad - \llbracket u_1 \oplus \sigma_1, \llbracket u_2 \oplus \sigma_2, u_3 \oplus \sigma_3 \rrbracket \rrbracket = -(\rho, \rho^*)(k) \oplus k,
\end{aligned}$$

with

$$k := [R_\Delta^{\text{bas}}(a_1, a_2)u_3 - R_\Delta(u_1, u_2)\sigma_3] + \text{c.p.}$$

To see that k is a section of K , recall from Theorem 5.10 that since (K, U, Δ) is Dirac and $u_i = (X_i, \xi_i) \in \Gamma(U)$, $i = 1, 2, 3$, we have $R_\Delta(u_1, u_2)\sigma_3 + \text{c.p.} \in \Gamma(K)$. By the same theorem, we find $R_\Delta^{\text{bas}}(a_1, a_2)u_3 + \text{c.p.} \in \Gamma(K)$ because (K, U, Δ) is an \mathcal{LA} -triple. We have, writing $\sigma_i = (a_i, \theta_i)$ for $i = 1, 2, 3$:

$$\begin{aligned}
& \llbracket \llbracket u_1 \oplus \sigma_1, u_2 \oplus \sigma_2 \rrbracket, u_3 \oplus \sigma_3 \rrbracket \\
&= \llbracket (\llbracket u_1, u_2 \rrbracket_\Delta + \nabla_{\sigma_1}^{\text{bas}} u_2 - \nabla_{\sigma_2}^{\text{bas}} u_1) \oplus ([\sigma_1, \sigma_2]_D + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, u_2 \rangle)), u_3 \oplus \sigma_3 \rrbracket \\
&= \left(\llbracket \llbracket u_1, u_2 \rrbracket_\Delta, u_3 \rrbracket_\Delta + \llbracket \nabla_{\sigma_1}^{\text{bas}} u_2 - \nabla_{\sigma_2}^{\text{bas}} u_1, u_3 \rrbracket_\Delta + \nabla_{[\sigma_1, \sigma_2]_D + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, u_2 \rangle)}^{\text{bas}} u_3 \right. \\
&\quad \left. - \nabla_{\sigma_3}^{\text{bas}} (\llbracket u_1, u_2 \rrbracket_\Delta + \nabla_{\sigma_1}^{\text{bas}} u_2 - \nabla_{\sigma_2}^{\text{bas}} u_1) \right) \\
&\oplus \left([[\sigma_1, \sigma_2]_D, \sigma_3]_D + [\Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1, \sigma_3]_D + [(0, \mathbf{d}\langle \sigma_1, u_2 \rangle), \sigma_3]_D \right. \\
&\quad \left. + \Delta_{\llbracket u_1, u_2 \rrbracket_\Delta + \nabla_{\sigma_1}^{\text{bas}} u_2 - \nabla_{\sigma_2}^{\text{bas}} u_1} \sigma_3 - \Delta_{u_3}([\sigma_1, \sigma_2]_D + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, u_2 \rangle)) \right. \\
&\quad \left. + (0, \mathbf{d}\langle [\sigma_1, \sigma_2]_D + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, u_2 \rangle), u_3 \rangle) \right).
\end{aligned}$$

But since $[(0, \mathbf{d}\langle \sigma_1, u_2 \rangle), \sigma_3]_D = 0$ and

$$\text{pr}_A([\sigma_1, \sigma_2]_D + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, u_2 \rangle)) = [a_1, a_2] + \text{pr}_A(\Omega_{u_1} a_2 - \Omega_{u_2} a_1),$$

this leads to

$$\begin{aligned}
& \llbracket [u_1 \oplus \sigma_1, u_2 \oplus \sigma_2], u_3 \oplus \sigma_3 \rrbracket \\
&= \left(\llbracket [u_1, u_2]_\Delta, u_3 \rrbracket_\Delta + \llbracket \nabla_{\sigma_1}^{\text{bas}} u_2 - \nabla_{a_2}^{\text{bas}} u_1, u_3 \rrbracket_\Delta + \nabla_{[a_1, a_2] + \text{pr}_A(\Omega_{u_1} a_2 - \Omega_{u_2} a_1)}^{\text{bas}} u_3 \right. \\
&\quad \left. - \nabla_{a_3}^{\text{bas}} (\llbracket u_1, u_2 \rrbracket_\Delta + \nabla_{a_1}^{\text{bas}} u_2 - \nabla_{a_2}^{\text{bas}} u_1) \right) \\
&\oplus \left([\sigma_1, \sigma_2]_D, \sigma_3 \right)_D + [\Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1, \sigma_3]_D \\
&\quad + \Delta_{\llbracket u_1, u_2 \rrbracket_\Delta + \nabla_{a_1}^{\text{bas}} u_2 - \nabla_{a_2}^{\text{bas}} u_1} \sigma_3 - \Delta_{u_3} ([\sigma_1, \sigma_2]_D + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, u_2 \rangle)) \\
&\quad + (0, \mathbf{d}\langle [\sigma_1, \sigma_2]_D + \Delta_{u_1} \sigma_2 - \Delta_{u_2} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, u_2 \rangle), u_3 \rangle).
\end{aligned}$$

In the same manner, we get:

$$\begin{aligned}
& \llbracket u_2 \oplus \sigma_2, [u_1 \oplus \sigma_1, u_3 \oplus \sigma_3] \rrbracket \\
&= \left(\llbracket u_2, [u_1, u_3]_\Delta \rrbracket_\Delta + \llbracket u_2, \nabla_{a_1}^{\text{bas}} u_3 - \nabla_{a_3}^{\text{bas}} u_1 \rrbracket_\Delta + \nabla_{a_2}^{\text{bas}} (\llbracket u_1, u_3 \rrbracket_\Delta + \nabla_{a_1}^{\text{bas}} u_3 - \nabla_{a_3}^{\text{bas}} u_1) \right. \\
&\quad \left. - \nabla_{[a_1, a_3] + \text{pr}_A(\Omega_{u_1} a_3 - \Omega_{u_3} a_1)}^{\text{bas}} u_2 \right) \\
&\oplus \left([\sigma_2, [\sigma_1, \sigma_3]_D]_D + [\sigma_2, \Delta_{u_1} \sigma_3 - \Delta_{u_3} \sigma_1]_D + [\sigma_2, (0, \mathbf{d}\langle \sigma_1, u_3 \rangle)]_D \right. \\
&\quad + \Delta_{u_2} ([\sigma_1, \sigma_3]_D + \Delta_{u_1} \sigma_3 - \Delta_{u_3} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, u_3 \rangle)) - \Delta_{\llbracket u_1, u_3 \rrbracket_\Delta + \nabla_{a_1}^{\text{bas}} u_3 - \nabla_{a_3}^{\text{bas}} u_1} \sigma_2 \\
&\quad \left. + (0, \mathbf{d}\langle \sigma_2, \llbracket u_1, u_3 \rrbracket_\Delta + \nabla_{a_1}^{\text{bas}} u_3 - \nabla_{a_3}^{\text{bas}} u_1 \rangle) \right).
\end{aligned}$$

We start by studying more carefully the U -part of $\text{Jac}_{[\cdot, \cdot]_\Delta}(u_1 \oplus \sigma_1, u_2 \oplus \sigma_2, u_3 \oplus \sigma_3)$. By the computations above, this equals

$$\begin{aligned}
& \llbracket [u_1, u_2]_\Delta, u_3 \rrbracket_\Delta + \llbracket u_2, [u_1, u_3]_\Delta \rrbracket_\Delta - \llbracket u_1, [u_2, u_3]_\Delta \rrbracket_\Delta \\
&\quad - R_{\nabla^{\text{bas}}}(a_1, a_2)u_3 + R_{\nabla^{\text{bas}}}(a_1, a_3)u_2 - R_{\nabla^{\text{bas}}}(a_2, a_3)u_1 \\
&\quad + \llbracket \nabla_{a_1}^{\text{bas}} u_2 - \nabla_{a_2}^{\text{bas}} u_1, u_3 \rrbracket_\Delta + \nabla_{\text{pr}_A(\Omega_{u_1} a_2 - \Omega_{u_2} a_1)}^{\text{bas}} u_3 - \nabla_{a_3}^{\text{bas}} \llbracket u_1, u_2 \rrbracket_\Delta \\
&\quad + \llbracket u_2, \nabla_{a_1}^{\text{bas}} u_3 - \nabla_{a_3}^{\text{bas}} u_1 \rrbracket_\Delta + \nabla_{a_2}^{\text{bas}} \llbracket u_1, u_3 \rrbracket_\Delta - \nabla_{\text{pr}_A(\Omega_{u_1} a_3 - \Omega_{u_3} a_1)}^{\text{bas}} u_2 \\
&\quad - \llbracket u_1, \nabla_{a_2}^{\text{bas}} u_3 - \nabla_{a_3}^{\text{bas}} u_2 \rrbracket_\Delta - \nabla_{a_1}^{\text{bas}} \llbracket u_2, u_3 \rrbracket_\Delta + \nabla_{\text{pr}_A(\Omega_{u_2} a_3 - \Omega_{u_3} a_2)}^{\text{bas}} u_1
\end{aligned}$$

Since $(U, \text{pr}_{TM}, [\cdot, \cdot]_\Delta)$ is a Lie algebroid, the first three terms cancel. By Theorem 6.1, Proposition 5.6, the fact that $\nabla_a^{\text{bas}} u \in \Gamma(U)$ for all $u \in \Gamma(U)$ and $a \in \Gamma(A)$, and the equality $\text{pr}_A \Omega_{u_1} a_2 = \text{pr}_A \Delta_{u_1} \sigma_2$, etc, we get hence that the U -part of $\text{Jac}_{[\cdot, \cdot]_\Delta}(u_1 \oplus \sigma_1, u_2 \oplus \sigma_2, u_3 \oplus \sigma_3)$ is

$$\begin{aligned}
& -(\rho, \rho^*)(R_\Delta^{\text{bas}}(a_1, a_2)u_3 + R_\Delta^{\text{bas}}(a_3, a_1)u_2 + R_\Delta^{\text{bas}}(a_2, a_3)u_1) \\
& + (\rho, \rho^*)(R_\Delta(u_1, u_2)\sigma_3 + R_\Delta(u_2, u_3)\sigma_1 + R_\Delta(u_3, u_1)\sigma_2).
\end{aligned}$$

We now look more carefully at the $A \oplus T^*M$ -part of $\text{Jac}[\cdot, \cdot]_\Delta(u_1 \oplus \sigma_1, u_2 \oplus \sigma_2, u_3 \oplus \sigma_3)$. By the formulas above, it equals

$$\begin{aligned}
& [[\sigma_1, \sigma_2]_D, \sigma_3]_D + [\Delta_{u_1}\sigma_2 - \Delta_{u_2}\sigma_1, \sigma_3]_D \\
& + \Delta_{[u_1, u_2]_\Delta + \nabla_{a_1}^{\text{bas}}u_2 - \nabla_{a_2}^{\text{bas}}u_1}\sigma_3 - \Delta_{u_3}([\sigma_1, \sigma_2]_D + \Delta_{u_1}\sigma_2 - \Delta_{u_2}\sigma_1 + (0, \mathbf{d}\langle\sigma_1, u_2\rangle)) \\
& + (0, \mathbf{d}\langle[\sigma_1, \sigma_2]_D + \Delta_{u_1}\sigma_2 - \Delta_{u_2}\sigma_1 + (0, \mathbf{d}\langle\sigma_1, u_2\rangle), u_3\rangle) \\
& + [\sigma_2, [\sigma_1, \sigma_3]_D]_D + [\sigma_2, \Delta_{u_1}\sigma_3 - \Delta_{u_3}\sigma_1]_D + [\sigma_2, (0, \mathbf{d}\langle\sigma_1, u_3\rangle)]_D \\
& + \Delta_{u_2}([\sigma_1, \sigma_3]_D + \Delta_{u_1}\sigma_3 - \Delta_{u_3}\sigma_1 + (0, \mathbf{d}\langle\sigma_1, u_3\rangle)) - \Delta_{[u_1, u_3]_\Delta + \nabla_{a_1}^{\text{bas}}u_3 - \nabla_{a_3}^{\text{bas}}u_1}\sigma_2 \\
& + (0, \mathbf{d}\langle\sigma_2, [u_1, u_3]_\Delta + \nabla_{a_1}^{\text{bas}}u_3 - \nabla_{a_3}^{\text{bas}}u_1\rangle) \\
& - [\sigma_1, [\sigma_2, \sigma_3]_D]_D - [\sigma_1, \Delta_{u_2}\sigma_3 - \Delta_{u_3}\sigma_2]_D - [\sigma_1, (0, \mathbf{d}\langle\sigma_2, u_3\rangle)]_D \\
& - \Delta_{u_1}([\sigma_2, \sigma_3]_D + \Delta_{u_2}\sigma_3 - \Delta_{u_3}\sigma_2 + (0, \mathbf{d}\langle\sigma_2, u_3\rangle)) + \Delta_{[u_2, u_3]_\Delta + \nabla_{a_2}^{\text{bas}}u_3 - \nabla_{a_3}^{\text{bas}}u_2}\sigma_1 \\
& - (0, \mathbf{d}\langle\sigma_1, [u_2, u_3]_\Delta + \nabla_{a_2}^{\text{bas}}u_3 - \nabla_{a_3}^{\text{bas}}u_2\rangle).
\end{aligned}$$

Sorting out the terms yields

$$\begin{aligned}
& [[\sigma_1, \sigma_2]_D, \sigma_3]_D + [\sigma_2, [\sigma_1, \sigma_3]_D]_D - [\sigma_1, [\sigma_2, \sigma_3]_D]_D \\
& - \Delta_{u_1}[\sigma_2, \sigma_3]_D + [\Delta_{u_1}\sigma_2, \sigma_3]_D + [\sigma_2, \Delta_{u_1}\sigma_3]_D + \Delta_{\nabla_{a_3}^{\text{bas}}u_1}\sigma_2 - \Delta_{\nabla_{a_2}^{\text{bas}}u_1}\sigma_3 - (0, \mathbf{d}\langle\sigma_2, \nabla_{a_3}^{\text{bas}}u_1\rangle) \\
& + \Delta_{u_2}[\sigma_1, \sigma_3]_D - [\Delta_{u_2}\sigma_1, \sigma_3]_D - [\sigma_1, \Delta_{u_2}\sigma_3]_D - \Delta_{\nabla_{a_3}^{\text{bas}}u_2}\sigma_1 + \Delta_{\nabla_{a_1}^{\text{bas}}u_2}\sigma_3 + (0, \mathbf{d}\langle\sigma_1, \nabla_{a_3}^{\text{bas}}u_2\rangle) \\
& - \Delta_{u_3}[\sigma_1, \sigma_2]_D - [\sigma_2, \Delta_{u_3}\sigma_1]_D + [\sigma_1, \Delta_{u_3}\sigma_2]_D + \Delta_{\nabla_{a_2}^{\text{bas}}u_3}\sigma_1 - \Delta_{\nabla_{a_1}^{\text{bas}}u_3}\sigma_2 - (0, \mathbf{d}\langle\sigma_1, \nabla_{a_2}^{\text{bas}}u_3\rangle) \\
& - R_\Delta(u_1, u_2)\sigma_3 - R_\Delta(u_2, u_3)\sigma_1 - R_\Delta(u_3, u_1)\sigma_2 \\
& - \Delta_{u_3}(0, \mathbf{d}\langle\sigma_1, u_2\rangle) + (0, \mathbf{d}\langle[\sigma_1, \sigma_2]_D + \Delta_{u_1}\sigma_2 - \Delta_{u_2}\sigma_1 + (0, \mathbf{d}\langle\sigma_1, u_2\rangle), u_3\rangle) \\
& + [\sigma_2, (0, \mathbf{d}\langle\sigma_1, u_3\rangle)]_D + \Delta_{u_2}(0, \mathbf{d}\langle\sigma_1, u_3\rangle) + (0, \mathbf{d}\langle\sigma_2, [u_1, u_3]_\Delta + \nabla_{a_1}^{\text{bas}}u_3\rangle) \\
& - [\sigma_1, (0, \mathbf{d}\langle\sigma_2, u_3\rangle)]_D - \Delta_{u_1}(0, \mathbf{d}\langle\sigma_2, u_3\rangle) - (0, \mathbf{d}\langle\sigma_1, [u_2, u_3]_\Delta\rangle).
\end{aligned}$$

By using the Jacobi identity for $[\cdot, \cdot]_D$ and three times (2) of Theorem 6.1 together with

$$-[\sigma_2, \Delta_{u_3}\sigma_1]_D = [\Delta_{u_3}\sigma_1, \sigma_2]_D - (0, \mathbf{d}\langle(\rho, \rho^*)\Delta_{u_3}\sigma_1, \sigma_2\rangle),$$

we get

$$\begin{aligned}
& R_\Delta^{\text{bas}}(a_2, a_3)u_1 - R_\Delta^{\text{bas}}(a_1, a_3)u_2 + R_\Delta^{\text{bas}}(a_1, a_2)u_3 \\
& - R_\Delta(u_1, u_2)\sigma_3 - R_\Delta(u_2, u_3)\sigma_1 - R_\Delta(u_3, u_1)\sigma_2 \\
& - \Delta_{u_3}(0, \mathbf{d}\langle\sigma_1, u_2\rangle) + (0, \mathbf{d}\langle[\sigma_1, \sigma_2]_D + \Delta_{u_1}\sigma_2 - \Delta_{u_2}\sigma_1 + (0, \mathbf{d}\langle\sigma_1, u_2\rangle), u_3\rangle) \\
& + [\sigma_2, (0, \mathbf{d}\langle\sigma_1, u_3\rangle)]_D + \Delta_{u_2}(0, \mathbf{d}\langle\sigma_1, u_3\rangle) + (0, \mathbf{d}\langle\sigma_2, [u_1, u_3]_\Delta + \nabla_{a_1}^{\text{bas}}u_3\rangle) \\
& - [\sigma_1, (0, \mathbf{d}\langle\sigma_2, u_3\rangle)]_D - \Delta_{u_1}(0, \mathbf{d}\langle\sigma_2, u_3\rangle) - (0, \mathbf{d}\langle\sigma_1, [u_2, u_3]_\Delta\rangle) - (0, \mathbf{d}\langle(\rho, \rho^*)\Delta_{u_3}\sigma_1, \sigma_2\rangle) \\
& = R_\Delta^{\text{bas}}(a_2, a_3)u_1 + R_\Delta^{\text{bas}}(a_3, a_1)u_2 + R_\Delta^{\text{bas}}(a_1, a_2)u_3 \\
& - R_\Delta(u_1, u_2)\sigma_3 - R_\Delta(u_2, u_3)\sigma_1 - R_\Delta(u_3, u_1)\sigma_2 + (0, \mathbf{d}\varphi),
\end{aligned}$$

with

$$\begin{aligned}
\varphi &= -X_3\langle\sigma_1, u_2\rangle + \langle[\sigma_1, \sigma_2]_D + \Delta_{u_1}\sigma_2 - \Delta_{u_2}\sigma_1 + (0, \mathbf{d}\langle\sigma_1, u_2\rangle), u_3\rangle + \rho(a_2)\langle\sigma_1, u_3\rangle + X_2\langle\sigma_1, u_3\rangle \\
&\quad + \langle\sigma_2, \llbracket u_1, u_3 \rrbracket_\Delta + \nabla_{a_1}^{\text{bas}}u_3\rangle - \rho(a_1)\langle\sigma_2, u_3\rangle - X_1\langle\sigma_2, u_3\rangle - \langle\sigma_1, \llbracket u_2, u_3 \rrbracket_\Delta\rangle - \langle(\rho, \rho^*)\Delta_{u_3}\sigma_1, \sigma_2\rangle \\
&= \langle[\sigma_1, \sigma_2]_D, u_3\rangle + \rho(a_2)\langle\sigma_1, u_3\rangle + \langle\sigma_2, \nabla_{a_1}^{\text{bas}}u_3\rangle - \rho(a_1)\langle\sigma_2, u_3\rangle - \langle(\rho, \rho^*)\Delta_{u_3}\sigma_1, \sigma_2\rangle \\
&\stackrel{(5.15)}{=} \langle-\nabla_{\sigma_2}^{\text{bas}}\sigma_1 + \Delta_{(\rho, \rho^*)\sigma_1}\sigma_2, u_3\rangle + \rho(a_2)\langle\sigma_1, u_3\rangle + \langle\sigma_2, \nabla_{a_1}^{\text{bas}}u_3 - (\rho, \rho^*)\Delta_{u_3}\sigma_1\rangle - \rho(a_1)\langle\sigma_2, u_3\rangle \\
&= -\langle\nabla_{\sigma_2}^{\text{bas}}\sigma_1, u_3\rangle + \rho(a_2)\langle\sigma_1, u_3\rangle + \langle\sigma_2, \nabla_{a_1}^{\text{bas}}u_3 - (\rho, \rho^*)\Delta_{u_3}\sigma_1 - \llbracket(\rho, \rho^*)\sigma_1, u_3\rrbracket_\Delta\rangle \\
&\stackrel{(5.16)}{=} -\langle\nabla_{\sigma_2}^{\text{bas}}\sigma_1, u_3\rangle + \rho(a_2)\langle\sigma_1, u_3\rangle - \langle\nabla_{\sigma_2}^{\text{bas}}u_3, \sigma_1\rangle - \langle\text{Skew}_\Delta(u_3, (\rho, \rho^*)\sigma_1), a_2\rangle \\
&\stackrel{(5.11)}{=} 0.
\end{aligned}$$

For the second equality, we have used $\langle(0, \mathbf{d}\langle\sigma_1, u_2\rangle), u_3\rangle = X_3\langle\sigma_1, u_2\rangle$ and two times the duality equation for Δ and $\llbracket \cdot, \cdot \rrbracket_\Delta$.

□

We have used Theorem 6.1 for the Jacobi identity. For completeness, we prove this theorem here.

Proof of Theorem 6.1. (1) For simplicity, we will write here a for $(a, 0)$, $\Delta_u a := \Delta_u(a, 0)$ and $\mathbf{d}\varphi := (0, \mathbf{d}\varphi)$ for $a \in \Gamma(A)$, $\varphi \in C^\infty(M)$ and $u \in \Gamma(U)$. First, we find that for $a, b \in \Gamma(A)$ and $u = (X, \xi), v = (Y, \eta) \in \Gamma(U)$, the equation

$$0 = \langle R_\Delta^{\text{bas}}(a, b)u, v \rangle$$

can be written

$$\begin{aligned}
0 &= -\langle \Delta_u[a, b] - \mathbf{d}\langle \xi, [a, b] \rangle, v \rangle + \rho(a)\langle \Delta_u b - \mathbf{d}\langle \xi, b \rangle, v \rangle - \langle \Delta_u b - \mathbf{d}\langle \xi, b \rangle, \mathcal{L}_a v \rangle \\
&\quad - \rho(b)\langle \Delta_u a - \mathbf{d}\langle \xi, a \rangle, v \rangle + \langle \Delta_u a - \mathbf{d}\langle \xi, a \rangle, \mathcal{L}_b v \rangle \\
&\quad + \langle \Delta_{(\rho, \rho^*)\Omega_u b + \mathcal{L}_b u} a - \mathbf{d}\langle (\rho, \rho^*)\Omega_u b + \mathcal{L}_b u, a \rangle, v \rangle \\
&\quad - \langle \Delta_{(\rho, \rho^*)\Omega_u a + \mathcal{L}_a u} b - \mathbf{d}\langle (\rho, \rho^*)\Omega_u a + \mathcal{L}_a u, b \rangle, v \rangle \\
&= -\langle \Delta_u[a, b], v \rangle + Y\langle \xi, [a, b] \rangle + \rho(a)\langle \Delta_u b, v \rangle - \rho(a)Y\langle \xi, b \rangle - \langle \Delta_u b, \mathcal{L}_a v \rangle + [\rho(a), Y]\langle \xi, b \rangle \\
&\quad - \rho(b)\langle \Delta_u a, v \rangle + \rho(b)Y\langle \xi, a \rangle + \langle \Delta_u a, \mathcal{L}_b v \rangle - [\rho(b), Y]\langle \xi, a \rangle \\
&\quad + \langle \Delta_{(\rho, \rho^*)\Omega_u b} a, v \rangle + [\rho(b), X]\langle a, \eta \rangle - \langle a, \llbracket \mathcal{L}_b u, v \rrbracket_\Delta \rangle - Y\langle (\rho, \rho^*)\Omega_u b + \mathcal{L}_b u, a \rangle \\
&\quad - \langle \Delta_{(\rho, \rho^*)\Omega_u a} b, v \rangle - [\rho(a), X]\langle b, \eta \rangle + \langle b, \llbracket \mathcal{L}_a u, v \rrbracket_\Delta \rangle + Y\langle (\rho, \rho^*)\Omega_u a + \mathcal{L}_a u, b \rangle.
\end{aligned}$$

We use

$$\begin{aligned}
&Y\langle \xi, [a, b] \rangle + \rho(b)Y\langle \xi, a \rangle - [\rho(b), Y]\langle \xi, a \rangle - Y\langle \mathcal{L}_b u, a \rangle \\
&= Y(-\langle \xi, [b, a] \rangle + \rho(b)\langle \xi, a \rangle - \langle \mathcal{L}_b \xi, a \rangle) = 0,
\end{aligned}$$

$$\begin{aligned}
&-\rho(a)Y\langle \xi, b \rangle + [\rho(a), Y]\langle \xi, b \rangle + Y\langle \mathcal{L}_a u, b \rangle \\
&= -Y\rho(a)\langle \xi, b \rangle + Y\langle \mathcal{L}_a \xi, b \rangle = -Y\langle \xi, [a, b] \rangle
\end{aligned}$$

and, by (5.15) and (5.11),

$$\begin{aligned}
& \langle \Delta_{(\rho, \rho^*)\Omega_u b} a, v \rangle - Y \langle (\rho, \rho^*)\Omega_u b, a \rangle \\
&= \langle \nabla_a^{\text{bas}} \Omega_u b + [\Omega_u b, a]_D, v \rangle - Y \langle (\rho, \rho^*)\Omega_u b, a \rangle \\
&= \rho(a) \langle \Omega_u b, v \rangle - \langle \Omega_u b, \nabla_a^{\text{bas}} v \rangle - \langle \text{Skew}_\Delta(v, (\rho, \rho^*)\Omega_u b), a \rangle - \langle [a, \Omega_u b]_D, v \rangle \\
&= \rho(a) \langle \Omega_u b, v \rangle - \langle \Omega_u b, \nabla_a^{\text{bas}} v \rangle - \langle \text{Skew}_\Delta(v, (\rho, \rho^*)\Omega_u b), a \rangle - \langle \mathcal{L}_a \Omega_u b, v \rangle \\
&= - \langle \Omega_u b, (\rho, \rho^*)\Omega_v a \rangle - \langle \text{Skew}_\Delta(v, (\rho, \rho^*)\Omega_u b), a \rangle
\end{aligned}$$

to get

$$\begin{aligned}
0 &= -Y \langle \xi, [a, b] \rangle - X \langle \eta, [a, b] \rangle + \langle [a, b], \llbracket u, v \rrbracket_\Delta \rangle - \langle a, \llbracket \mathcal{L}_b u, v \rrbracket_\Delta \rangle + \langle b, \llbracket \mathcal{L}_a u, v \rrbracket_\Delta \rangle \\
&\quad + \rho(a) \langle \Delta_u b, v \rangle - X \langle b, \mathcal{L}_a v \rangle + \langle b, \llbracket u, \mathcal{L}_a v \rrbracket_\Delta \rangle \\
&\quad - \rho(b) \langle \Delta_u a, v \rangle + X \langle a, \mathcal{L}_b v \rangle - \langle a, \llbracket u, \mathcal{L}_b v \rrbracket_\Delta \rangle \\
&\quad + [\rho(b), X] \langle a, \eta \rangle - [\rho(a), X] \langle b, \eta \rangle \\
&\quad - \langle \Omega_u b, (\rho, \rho^*)\Omega_v a \rangle - \langle \text{Skew}_\Delta(v, (\rho, \rho^*)\Omega_u b), a \rangle \\
&\quad + \langle \Omega_u a, (\rho, \rho^*)\Omega_v b \rangle + \langle \text{Skew}_\Delta(v, (\rho, \rho^*)\Omega_u a), b \rangle.
\end{aligned}$$

But since

$$\begin{aligned}
\rho(a) \langle \Delta_u b, v \rangle - [\rho(a), X] \langle b, \eta \rangle - X \langle b, \mathcal{L}_a v \rangle &= -\rho(a) \langle b, \llbracket u, v \rrbracket_\Delta \rangle + X \langle [a, b], \eta \rangle \\
&= -\langle [a, b], \llbracket u, v \rrbracket_\Delta \rangle - \langle b, \mathcal{L}_a \llbracket u, v \rrbracket_\Delta \rangle + X \langle [a, b], \eta \rangle,
\end{aligned}$$

we find the equation

$$\begin{aligned}
0 &= -Y \langle \xi, [a, b] \rangle + X \langle \eta, [a, b] \rangle - \langle [a, b], \llbracket u, v \rrbracket_\Delta \rangle - \langle a, \llbracket \mathcal{L}_b u, v \rrbracket_\Delta \rangle + \langle b, \llbracket \mathcal{L}_a u, v \rrbracket_\Delta \rangle \\
&\quad + \langle b, \llbracket u, \mathcal{L}_a v \rrbracket_\Delta \rangle - \langle a, \llbracket u, \mathcal{L}_b v \rrbracket_\Delta \rangle - \langle b, \mathcal{L}_a \llbracket u, v \rrbracket_\Delta \rangle + \langle a, \mathcal{L}_b \llbracket u, v \rrbracket_\Delta \rangle \\
&\quad - \langle \Omega_u b, (\rho, \rho^*)\Omega_v a \rangle - \langle \text{Skew}_\Delta(v, (\rho, \rho^*)\Omega_u b), a \rangle \\
(5.20) \quad &+ \langle \Omega_u a, (\rho, \rho^*)\Omega_v b \rangle + \langle \text{Skew}_\Delta(v, (\rho, \rho^*)\Omega_u a), b \rangle.
\end{aligned}$$

Now, writing $\sigma = (a, \theta)$, we compute:

$$\begin{aligned}
& \nabla_\sigma^{\text{bas}} \llbracket u, v \rrbracket_\Delta - \llbracket \nabla_\sigma^{\text{bas}} u, v \rrbracket_\Delta - \llbracket u, \nabla_\sigma^{\text{bas}} v \rrbracket_\Delta + \nabla_{\Delta_u \sigma}^{\text{bas}} v - \nabla_{\Delta_v \sigma}^{\text{bas}} u \\
&= (\rho, \rho^*) \Omega_{\llbracket u, v \rrbracket_\Delta} a + \mathcal{L}_a \llbracket u, v \rrbracket_\Delta - \llbracket (\rho, \rho^*) \Omega_u a + \mathcal{L}_a u, v \rrbracket_\Delta - \llbracket u, (\rho, \rho^*) \Omega_v a + \mathcal{L}_a v \rrbracket_\Delta \\
&\quad + \nabla_{\text{pr}_A}^{\text{bas}} \Omega_u a v - \nabla_{\text{pr}_A}^{\text{bas}} \Omega_v a u \\
&\stackrel{(5.16)}{=} (\rho, \rho^*) (\Delta_{\llbracket u, v \rrbracket_\Delta} a - \mathbf{d} \langle a, \llbracket u, v \rrbracket_\Delta \rangle) + \mathcal{L}_a \llbracket u, v \rrbracket_\Delta - \llbracket \mathcal{L}_a u, v \rrbracket_\Delta - \llbracket u, \mathcal{L}_a v \rrbracket_\Delta \\
&\quad - (\rho, \rho^*) \Delta_u \Omega_v a + \langle \nabla_{\cdot}^{\text{bas}} u, \Omega_v a \rangle + (\rho, \rho^*) \Delta_v \Omega_u a - \langle \nabla_{\cdot}^{\text{bas}} v, \Omega_u a \rangle \\
&\quad - \text{Skew}_\Delta((\rho, \rho^*) \Omega_u a, v) \\
&= -(\rho, \rho^*) R_\Delta(u, v) a - \rho^* \mathbf{d} \langle a, \llbracket u, v \rrbracket_\Delta \rangle + \mathcal{L}_a \llbracket u, v \rrbracket_\Delta - \llbracket \mathcal{L}_a u, v \rrbracket_\Delta - \llbracket u, \mathcal{L}_a v \rrbracket_\Delta \\
&\quad + \rho^* \mathbf{d} \langle X \langle \eta, a \rangle \rangle + \langle \nabla_{\cdot}^{\text{bas}} u, \Omega_v a \rangle - \rho^* \mathbf{d} \langle Y \langle \xi, a \rangle \rangle - \langle \nabla_{\cdot}^{\text{bas}} v, \Omega_u a \rangle \\
&\quad - \text{Skew}_\Delta((\rho, \rho^*) \Omega_u a, v).
\end{aligned}$$

We write the right-hand side of this equation

$$-(\rho, \rho^*) R_\Delta(u, v) a + w$$

with $w \in \Gamma(TM \oplus A^*)$. Since

$$\text{pr}_{TM}(\mathcal{L}_a \llbracket u, v \rrbracket_\Delta - \llbracket \mathcal{L}_a u, v \rrbracket_\Delta - \llbracket u, \mathcal{L}_a v \rrbracket_\Delta) = [\rho(a), [X, Y]] - [[\rho(a), X], Y] - [X, [\rho(a), Y]]$$

vanishes, we observe that $\text{pr}_{TM} w = 0$. Hence, we just have to show that $\langle w, b \rangle = 0$ for any section $b \in \Gamma(A)$. We have

$$\begin{aligned}
\langle w, b \rangle &= \langle \mathcal{L}_a \llbracket u, v \rrbracket_\Delta - \llbracket \mathcal{L}_a u, v \rrbracket_\Delta - \llbracket u, \mathcal{L}_a v \rrbracket_\Delta, b \rangle \\
&\quad - \rho(b) \langle a, \llbracket u, v \rrbracket_\Delta \rangle + \rho(b) X \langle \eta, a \rangle - \rho(b) Y \langle \xi, a \rangle + \langle \nabla_b^{\text{bas}} u, \Omega_v a \rangle - \langle \nabla_b^{\text{bas}} v, \Omega_u a \rangle \\
&\quad - \langle \text{Skew}_\Delta((\rho, \rho^*) \Omega_u a, v), b \rangle \\
&= \langle \mathcal{L}_a \llbracket u, v \rrbracket_\Delta - \llbracket \mathcal{L}_a u, v \rrbracket_\Delta - \llbracket u, \mathcal{L}_a v \rrbracket_\Delta, b \rangle \\
&\quad + \langle [a, b], \llbracket u, v \rrbracket_\Delta \rangle - \langle a, \mathcal{L}_b \llbracket u, v \rrbracket_\Delta \rangle + \rho(b) X \langle \eta, a \rangle - \rho(b) Y \langle \xi, a \rangle \\
&\quad + \langle (\rho, \rho^*) \Omega_u b + \mathcal{L}_b u, \Omega_v a \rangle - \langle (\rho, \rho^*) \Omega_v b + \mathcal{L}_b v, \Omega_u a \rangle \\
&\quad - \langle \text{Skew}_\Delta((\rho, \rho^*) \Omega_u a, v), b \rangle.
\end{aligned}$$

Since the definition of Ω and the duality of Δ and $\llbracket \cdot, \cdot \rrbracket_\Delta$ yield

$$\begin{aligned}
\langle \mathcal{L}_b u, \Omega_v a \rangle + \rho(b) X \langle \eta, a \rangle &= \langle \mathcal{L}_b u, \Delta_v a \rangle + X \rho(b) \langle \eta, a \rangle \\
&= Y \langle \mathcal{L}_b u, a \rangle - \langle \llbracket v, \mathcal{L}_b u \rrbracket_\Delta, a \rangle + X \rho(b) \langle \eta, a \rangle,
\end{aligned}$$

we get

$$\begin{aligned}
\langle w, b \rangle &= \langle \mathcal{L}_a \llbracket u, v \rrbracket_\Delta - \llbracket \mathcal{L}_a u, v \rrbracket_\Delta - \llbracket u, \mathcal{L}_a v \rrbracket_\Delta, b \rangle \\
&\quad + \langle [a, b], \llbracket u, v \rrbracket_\Delta \rangle - \langle a, \mathcal{L}_b \llbracket u, v \rrbracket_\Delta \rangle + \langle (\rho, \rho^*) \Omega_u b, \Omega_v a \rangle - \langle (\rho, \rho^*) \Omega_v b, \Omega_u a \rangle \\
&\quad - \langle \text{Skew}_\Delta((\rho, \rho^*) \Omega_u a, v), b \rangle \\
&\quad + Y \langle \mathcal{L}_b \xi, a \rangle - \langle \llbracket v, \mathcal{L}_b u \rrbracket_\Delta, a \rangle + X \rho(b) \langle \eta, a \rangle \\
&\quad - X \langle \mathcal{L}_b \eta, a \rangle + \langle \llbracket u, \mathcal{L}_b v \rrbracket_\Delta, a \rangle - Y \rho(b) \langle \xi, a \rangle \\
&= \langle \mathcal{L}_a \llbracket u, v \rrbracket_\Delta - \llbracket \mathcal{L}_a u, v \rrbracket_\Delta - \llbracket u, \mathcal{L}_a v \rrbracket_\Delta, b \rangle \\
&\quad + \langle [a, b], \llbracket u, v \rrbracket_\Delta \rangle - \langle a, \mathcal{L}_b \llbracket u, v \rrbracket_\Delta - \llbracket u, \mathcal{L}_b v \rrbracket_\Delta + \llbracket v, \mathcal{L}_b u \rrbracket_\Delta \rangle \\
&\quad + \langle (\rho, \rho^*) \Omega_u b, \Omega_v a \rangle - \langle (\rho, \rho^*) \Omega_v b, \Omega_u a \rangle \\
&\quad - \langle \text{Skew}_\Delta((\rho, \rho^*) \Omega_u a, v), b \rangle - X \langle \eta[a, b] \rangle + Y \langle \xi, [a, b] \rangle.
\end{aligned}$$

We have

$$-\langle a, \llbracket v, \mathcal{L}_b u \rrbracket_\Delta \rangle = \langle a, \llbracket \mathcal{L}_b u, v \rrbracket_\Delta \rangle - \langle \text{Skew}_\Delta(\mathcal{L}_b u, v), a \rangle$$

and since $\nabla_b^{\text{bas}} u \in \Gamma(U)$ by hypothesis, we find

$$-\text{Skew}_\Delta(\mathcal{L}_b u, v) = -\text{Skew}_\Delta(\mathcal{L}_b u - \nabla_b^{\text{bas}} u, v) = \text{Skew}_\Delta((\rho, \rho^*) \Omega_u b, v).$$

This shows that $\langle w, b \rangle$ is equal to the right-hand side of (5.20) and is hence vanishing.

(2) We compute, writing $u = (X, \xi)$ and $\sigma_i = (a_i, \theta_i)$ for $i = 1, 2$:

$$\begin{aligned}
& \Delta_u[\sigma_1, \sigma_2]_D - [\Delta_u \sigma_1, \sigma_2]_D - [\sigma_1, \Delta_u \sigma_2]_D + \Delta_{\nabla_{a_1}^{\text{bas}} u} \sigma_2 - \Delta_{\nabla_{a_2}^{\text{bas}} u} \sigma_1 + (0, \mathbf{d}\langle \sigma_1, \nabla_{a_2}^{\text{bas}} u \rangle) \\
&= \Omega_u[a_1, a_2] + (0, \mathcal{L}_X(\mathcal{L}_{\rho(a_1)} \theta_2 - \mathbf{i}_{\rho(a_2)} \mathbf{d}\theta_1) + \mathbf{d}\langle \xi, [a_1, a_2] \rangle) \\
&\quad - [\Omega_u a_1 + (0, \mathcal{L}_X \theta_1 + \mathbf{d}\langle \xi, a_1 \rangle), \sigma_2]_D - [\sigma_1, \Omega_u a_2 + (0, \mathcal{L}_X \theta_2 + \mathbf{d}\langle \xi, a_2 \rangle)]_D \\
&\quad + \Omega_{\nabla_{a_1}^{\text{bas}} u} a_2 + (0, \mathcal{L}_{\text{pr}_{TM}(\nabla_{a_1}^{\text{bas}} u)} \theta_2 + \mathbf{d}\langle \nabla_{a_1}^{\text{bas}} u, (a_2, 0) \rangle) \\
&\quad - \Omega_{\nabla_{a_2}^{\text{bas}} u} a_1 - (0, \mathcal{L}_{\text{pr}_{TM}(\nabla_{a_2}^{\text{bas}} u)} \theta_1 + \mathbf{d}\langle \nabla_{a_2}^{\text{bas}} u, (a_1, 0) \rangle) + (0, \mathbf{d}\langle \sigma_1, \nabla_{a_2}^{\text{bas}} u \rangle) \\
&= -R_{\Delta}^{\text{bas}}(a_1, a_2)u \\
&\quad + (0, \mathcal{L}_X(\mathcal{L}_{\rho(a_1)} \theta_2 - \mathbf{i}_{\rho(a_2)} \mathbf{d}\theta_1) + \mathbf{d}\langle \xi, [a_1, a_2] \rangle) \\
&\quad + (0, \mathbf{i}_{\rho(a_2)} \mathbf{d}\mathcal{L}_X \theta_1 - \mathcal{L}_{\rho \circ \text{pr}_A} \Omega_u a_1 \theta_2 - \mathbf{d}\langle (\rho, \rho^*) \Omega_u a_1, (a_2, 0) \rangle) \\
&\quad - (0, \mathcal{L}_{\rho(a_1)} \mathcal{L}_X \theta_2 + \mathcal{L}_{\rho(a_1)} \mathbf{d}\langle \xi, a_2 \rangle + \mathbf{i}_{\rho \circ \text{pr}_A} \Omega_u a_2 \mathbf{d}\theta_1) \\
&\quad + (0, \mathcal{L}_{\text{pr}_{TM}(\nabla_{a_1}^{\text{bas}} u)} \theta_2 + \mathbf{d}\langle \nabla_{a_1}^{\text{bas}} u, (a_2, 0) \rangle) \\
&\quad - (0, \mathcal{L}_{\text{pr}_{TM}(\nabla_{a_2}^{\text{bas}} u)} \theta_1 + \mathbf{d}\langle \nabla_{a_2}^{\text{bas}} u, (a_1, 0) \rangle) + (0, \mathbf{d}\langle \sigma_1, \nabla_{a_2}^{\text{bas}} u \rangle).
\end{aligned}$$

But we have

$$\begin{aligned}
& \mathcal{L}_X \mathcal{L}_{\rho(a_1)} \theta_2 - \mathcal{L}_{\rho \circ \text{pr}_A} \Omega_u a_1 \theta_2 - \mathcal{L}_{\rho(a_1)} \mathcal{L}_X \theta_2 + \mathcal{L}_{\text{pr}_{TM}(\nabla_{a_1}^{\text{bas}} u)} \theta_2 \\
&= \mathcal{L}_{[X, \rho(a_1)] - \rho \circ \text{pr}_A} \Omega_u a_1 + \mathcal{L}_{\text{pr}_{TM}(\nabla_{a_1}^{\text{bas}} u)} \theta_2 = 0
\end{aligned}$$

since $\text{pr}_{TM}(\nabla_{a_1}^{\text{bas}} u) = [\rho(a_1), X] + \rho \circ \text{pr}_A \Omega_u a_1$, and in the same manner

$$\begin{aligned}
& -\mathcal{L}_X \mathbf{i}_{\rho(a_2)} \mathbf{d}\theta_1 + \mathbf{i}_{\rho(a_2)} \mathbf{d}\mathcal{L}_X \theta_1 + \mathbf{i}_{\rho \circ \text{pr}_A} \Omega_u a_2 \mathbf{d}\theta_1 - \mathcal{L}_{\text{pr}_{TM}(\nabla_{a_2}^{\text{bas}} u)} \theta_1 \\
&\quad + \mathbf{d}\langle \nabla_{a_2}^{\text{bas}} u, (a_1, 0) \rangle - \mathbf{d}\langle \sigma_1, \nabla_{a_2}^{\text{bas}} u \rangle \\
&= \mathbf{i}_{[\rho(a_2), X] + \rho \circ \text{pr}_A} \Omega_u a_2 \mathbf{d}\theta_1 - \mathcal{L}_{\text{pr}_{TM}(\nabla_{a_2}^{\text{bas}} u)} \theta_1 - \mathbf{d}\langle (0, \theta_1), \nabla_{a_2}^{\text{bas}} u \rangle = 0.
\end{aligned}$$

Since

$$-\langle (\rho, \rho^*) \Omega_u a_1, (a_2, 0) \rangle + \langle \nabla_{a_1}^{\text{bas}} u, (a_2, 0) \rangle = \langle \mathcal{L}_{a_1} u, (a_2, 0) \rangle = \langle \mathcal{L}_{a_1} \xi, a_2 \rangle$$

and

$$\mathcal{L}_{\rho(a_1)} \mathbf{d}\langle \xi, a_2 \rangle = \mathbf{d}(\rho(a_1) \langle \xi, a_2 \rangle) = \mathbf{d}(\langle \mathcal{L}_{a_1} \xi, a_2 \rangle + \langle \xi, [a_1, a_2] \rangle),$$

the remaining sum

$$\mathbf{d}\langle \xi, [a_1, a_2] \rangle - \mathbf{d}\langle (\rho, \rho^*) \Omega_u a_1, (a_2, 0) \rangle - \mathcal{L}_{\rho(a_1)} \mathbf{d}\langle \xi, a_2 \rangle + \mathbf{d}\langle \nabla_{a_1}^{\text{bas}} u, (a_2, 0) \rangle$$

vanishes as well.

□